An Alternate Convergence Argument for Piecewise Smooth Functions

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1 Introduction

This is a past version of my paper, a newer version of which can be found at this link; the new version contains a more complete introduction as well as a more careful treatment of the details of the argument. I have nevertheless included this past version here, as it provides an alternative convergence argument to the Spectral Decomposition Theorem.

Before discussing the specifics of past results and our new technique, we will briefly define the Frobenius-Perron operator. If S is a function and $\mathcal{X}$ is a random variable with distribution function d, then the Frobenius-Perron operator associated to S is the operator

$$P : L^1 \to L^1$$

such that $P(d)$ is the distribution function for $S(\mathcal{X})$. If S is injective and differentiable, then

$$P(d)(t) = \left(S^{-1}\right)'(t) \cdot d\left(S^{-1}(t)\right)$$

is the Frobenius-Perron Operator.

In their book, Lasoto and Mackey were able to show that for a function $S : [0,1] \to [0,1]$ fulfilling the following properties its associated Frobenius-Perron operator P would be asymptotically periodic. That is, for a K sufficiently large, $P^{kn} f$ would converge for every f in L^1. Below are the conditions on S.

1. There is a collection of disjoint open intervals $\{I_i\}_{i=1}^{\overline{n}}$ whose closures cover $[0,1]$ such that for each integer $1 \leq i \leq \overline{n}$, the restriction of S to I_i is a C^2 function.

2. There exists some constant $\lambda > 1$ such that for every $1 \leq i \leq \overline{n}$ and $x \in I_i$, $|S'| \geq \lambda > 1$.

3. There is a real constant c such that

$$\frac{|S''(x)|}{|S'(x)|^2} \leq c < \infty\,.$$

Our simplest example of such an S would be $S(x) = rx + b \mod 1$ for $r > 1$, with representatives chosen in $[0, 1)$.

2 Reduction

Let $\{I_i\}_{i=0}^{\overline{n}}$ be a collection of disjoint bounded intervals that partition $[0, 1]$. Let $1 < r$ be a fixed real number. We let S be a function fulfilling the following conditions.

1. $S'(x)$ exists for every $1 \leq i \leq \overline{n}$ and $x \in I_i$.
2. For each i, $S'(x)$ is either nondecreasing on I_i or nonincreasing on I_i.
3. $r \leq |S'(x)|$ for all $x \in I_i$.
4. $S([0, 1]) \subset [0, 1]$

We will fix an S that fulfills these conditions for the rest of the paper. We will call it "nice."

Theorem 2.1. *If S fulfills the conditions above and P is the Frobenius-Perron operator associated to S, then there exist some functions $\{f_i\}_{i=0}^{k}$ such that for every non-negative $f \in L^1_{[0,1]}$, there exist some non-negative real numbers $\{\lambda_i\}_{i=0}^{k}$ so that*

$$\lim_{n\to\infty} P^n f = \sum_{i=0}^{k} \lambda_i f_i$$

in the L^1 sense. Further, each $\|f_i\|_{L^1} = 1$.

We claim that proving the following reduction is sufficient.

Theorem 2.2. *Let S fulfill the conditions above and P be the Frobenius-Perron operator associated to S. Let ϵ_S be some fixed constant depending on S. Then there exist some functions $\{f_i\}_{i=0}^{k}$ depending on S that fulfill the following. If I is an interval of length at most ϵ_S contained in some I_j, then there exist some non-negative real numbers $\{\lambda_i\}_{i=0}^{k}$ so that*

$$\lim_{n\to\infty} P^n 1_I = \sum_{i=0}^{k} \lambda_i f_i$$

in the L^1 sense. Further, each $\|f_i\|_{L^1} = 1$.

Proof that Theorem 1.2 implies Theorem 1.1. Fix an $f \in L^1_{[0,1]}$ with $\|f\|_{\mathrm{L}^1} = 1$. Note that we may write any step function as

$$\sum_{i=0}^{m} \alpha_i 1_{I_i}$$

where each $\alpha_i \in \mathbb{R}$ and I_i fulfills the conditions in Theorem 1.2 . Further, since step functions approximate measurable functions, there exist some sequence of such step functions φ_j that converge to f with each $\|\varphi_j\|_{\mathrm{L}^1} = 1$. Since P is linear, we may apply Theorem 1.2 to note that for some $\lambda_{i,j}$, we have that

$$\lim_{n\to\infty} P^n\varphi_j = \sum_{i=0}^{k} \lambda_{i,j} f_i \, .$$

Since for every $g \in L^1_{[0,1]}$, $\|P^n g\|_{\mathrm{L}^1} \leq \|g\|_{\mathrm{L}^1}$, it follows that each $\lambda_{i,j} \leq 1$. Then there exists some sub sequence j_l so that $\lambda_{i,j_l} \to \lambda_i$ as l goes to infinity for some λ_i. Then notice that

$$\limsup_{n\to\infty} \|P^n(\varphi_{j_l} - f\|_{\mathrm{L}^1} \leq \|P^n(\varphi_{j_l} - f\|_{\mathrm{L}^1} \ ,$$

which goes to 0 as l goes to infinity. The result follows. □

3 Construction

Since S is nice, it follows that the restriction $S_i = S|_{I_i} : I_i \to S(I_i)$ is a bijection. We use it to define an almost-inverse.

$$g_i(x) = \begin{cases} (S_i)^{-1}(x) & : x \in S_i(I_i) \\ 0 & : x \notin S_i(I_i) \end{cases}$$

We use our almost-inverses to define the Frobenius-Perron operator

$$\mathcal{P} : L^1([0,1]) \to L^1([0,1])$$

$$(\mathcal{P}(d))\,(x) = \sum_{i=0}^{\overline{n}} g_i'(x) \cdot d\,(g_i(x)) \ .$$

If $\mathcal{X}$ is a random variable with measurable density function d, then $S(\mathcal{X})$ has density function $\mathcal{P}(d)$.

Lemma 3.1. *If d is in $L^1([0,1])$, then*

$$\int |\mathcal{P}(d)| \leq \int |d| \ .$$

Further, if d is non-negative, then

$$\int \mathcal{P}(d) = \int d\,.$$

Additionally, we have that $\mathcal{P}$ can move inside of integrals according to the following lemma.

Lemma 3.2. *If $d \in L^1(S)$ is non-negative, and*

$$d(x) = \int J_t(x) dt \,,$$

for a family of functions J_t, then

$$\mathcal{P}(d)(x) = \int \mathcal{P}(J_t)(x) dx \,.$$

We then have Lemma 1.3, which allows us to usually reduce to working with indicator functions of intervals.

Lemma 3.3. *If $I \subset I_i$ is an interval, then so is $V(I)$. Further, if $0 < t$ and*

$$J_t = \{x : r\mathcal{P}(1_I)(x) > t\} \,,$$

then J_t is an interval as well.

Combining Lemmas 1.2 and 1.3, for every $n \in \mathbb{N}$, there should be some collection of intervals I_v indexed over $v \in [0,1]^n$ so that

$$\mathcal{P}^n d(x) = \int_{v \in [0,1]^n} 1_{I_v}(x) dv \,.$$

To construct this more exactly, it is useful to introduce definitions that let us work inductively on $[0,1]^n$ more easily. We begin with a simple definition of a k-nary tree, which we then expand to a "continuous" tree.

Definition 3.3.1. *We call T_k to be the k-nary tree and define it as the collection of all n-tuples of integers $(a_i)_{i=0}^n$ that fulfill the properties below.*

- $a_0 = 0$
- $0 \leq a_i < k$

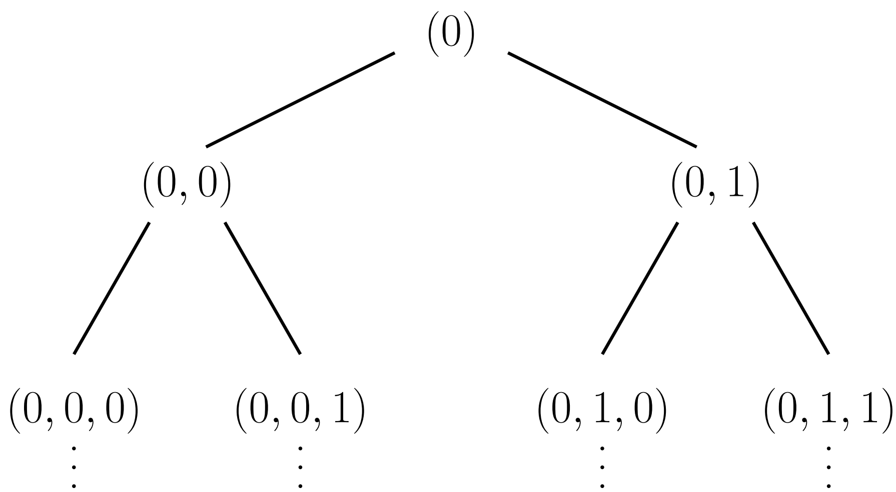
It is often easiest to write an element $e \in T_k$ explicitly as $e = (a_1, a_2, ...a_n)$. We will often denote the root element of the tree, the unique element of size 0, as (0). We recommend thinking of a_0 as representing beginning at the root of the tree and each a_i as the choice of children that leads to the current position. Below is a figure of the first nodes in the binary tree written in our notation.

Definition 3.3.2. *If $e \in T_k$, we call $|e|$ the size of e, where if $e = (a_i)_{i=0}^n$, then*

$$|e| = n$$

The choice to index from 0 instead of 1 will allow us to have powers $\mathring{e}$ in a natural way.

Definition 3.3.3. *If $e, f \in T_k$, $e = (a_i)_{i=0}^n$, and $f = (b_i)_{i=0}^m$, then we say $e \subseteq f$ if $n \leq m$ and $a_i = b_i$ for each $0 \leq i \leq n$*



Definition 3.3.4. *If $e \in T_k$ and $|e| > 0$, we define its parent $\overline{e}$ to be the unique element so that $\overline{e} \subset e$ and $|\overline{e}| + 1 = |e|$.*

Definition 3.3.5. *If $e, f \in T_k$, $e = (a_i)_{i=0}^{n}$, and $f = (b_i)_{i=0}^{m}$ then we say that their conjunction $ef \in T_k$ is $(c_i)_{i=0}^{n+m}$, where if $i \leq n$, $c_i = a_i$, and for $i > n$, $c_i = b_{i-n}$. It is helpful to note that written out,*

$$ef = (a_0, a_1, \ldots, a_n, b_1, \ldots, b_m)\,.$$

Effectively, a conjugation treats the node e as the root of the sub-tree and then chooses f in that sub-tree. This is why we have to remove the first 0 element from f.

Definition 3.3.6. *If $e \in T_k$, let $D^h(e) = \{f \in T_k : e \subseteq f, |f| = |e| + h\}$*

We will often write $D^1(e)$ simply as $D(e)$ and call them the descendants of e. Note that for all $f \in D(e)$, $\overline{f} = e$. We use similar notation for the continuous tree over $[0,1]$, which we denote $T_{[0,1]}$, the collection of all $n+1$ tuples of real numbers $(t_0, t_1, ..., t_n)$ that fulfill

- $t_0 = 0$
- $0 \leq t_i \leq 1$ for all $0 \leq i \leq n$.

For the sake of consistency, we will denote the elements of T_k with lower-case letters starting from e and the elements of $T_{[0,1]}$ with upper-case letters starting from E.

Definition 3.3.7. *Let $\mathcal{J}$ be a finite collection of intervals, where for each J_j there exists some i such that $J_j \subset I_i$. We say that an interval $I \neq \emptyset$ has a decomposition*

$$(T_k, T_{[0,1]}, I_{e,E}, c_{e,E,j})$$

by the collection $\mathcal{J}$ if it fulfills the following conditions. Firstly, $I_{e,E}$ are intervals whenever $|e| = |E|$. Additionally, $c_{e,E,j}$ are non-negative real numbers.

1.

$$\mathcal{P}(1_{I_{e,E}})(x) = \frac{1}{r}\left(\int_{F\in D(E)} \sum_{f\in D(e)} 1_{I_{f,F}}(x)dF\right) + \frac{1}{r}\sum_{j} c_{e,E,j}\cdot 1_{J_j}$$

2.

$$\sum_{e\in T_k} \frac{1}{r^{|e|}}\left(\int_{|E|=|e|} \left\|1_{I_{e,E})}\right\|_{L^1} dE\right) < \infty$$

3.

$$I_{(0),(0)} = I$$

Suppose that I has a decomposition by $\mathcal{J}$. Suppose further that each $J_\ell \in \mathcal{J}$ also has a decomposition by J; we will call them $(T_{k_j}, T_{[0,1]}, J_{j,e,C}, C_{\ell,e,E,j})$. Then it follows that there exists some collection of constants $a^n_{\ell,e,E}$ and $\alpha^n_{e,E}$ such that for every $n \in \mathbb{Z}^+$,

$$P^n\left(1_I\right) = \sum_{m=0}^{\infty}\sum_{|e|=m}\int_{|E|=m}\left(\alpha^n_{e,E}\cdot 1_{I_{e,E}} + \sum_{\ell} a^n_{\ell,e,E}\cdot 1_{J_{\ell,e,E}}\right)dE\,.$$

In other words, we can reduce from working with functions to a structured collection of real numbers; we will examine the particulars of this structure later.

Condition 2 allows us gives us decay in $|e|$, so that we can focus on showing that $\left|\alpha^n_{e,E}\right|$ and $\left|a^n_{\ell,e,E}\right|$ converge when $|e|$ is small. From this, the convergence of everything shall follows. Our final condition simply ensures that we are in fact decomposing I, and not some other object.

Definition 3.3.8. *Suppose that J is an interval entirely contained in some I_i. We say that J **does not split** under S^n if there exist some $\{j_m\}_{m=1}^n$ such that each $S^m(J) \subset I_{j_m}$, and thus each $S^m(J)$ is an interval.*

Definition 3.3.9. *We say that $\epsilon_0 > 0$ is (i, N) **admissible** if neither $(a_i, a_i+\epsilon)$ nor $(b_i - \epsilon, b_i)$ splits under S^N.*

Lemma 3.4. *For every i and N, an (i, N) admissible ϵ_i exists.*

Proof. Note that $[a_i, a_i + \epsilon]$ is an interval entirely contained within I_i. Then $S([a_i, a_i + \epsilon]) \subset [0, 1]$ is an interval. Then we can choose ϵ small enough that $(a_i, a_i + \epsilon) \subset I_{j_1}$ for some j_1. As usual, we ignore what occurs at endpoints. Repeating this process N times, choosing ϵ smaller at each step, we arrive at some ϵ that suffices. Repeating this for $[b_i - \epsilon, b_i]$, we conclude that an ϵ_i exists. $\square$

Lemma 3.5. *If $0 < \epsilon_0$ is (i, N) admissible, and $0 < \epsilon < \epsilon_0$, then ϵ is (i, N) admissible as well.*

If $r > 2\overline{n}$, then splitting is unimportant, as we will get geometric decay that handles Condition 2 of being a decomposition. If $r < 2\overline{n}$, we need to ensure that splitting does not occur frequently. From here on we fix N to be large enough that $r^N > 2\overline{n} + 1$, and now define the collection for our decompositions. Note that $\overline{n} > 1$ necessarily, and so $r^N > 5$. We also let $l = \inf\{b_i - a_i\}$.

It will be helpful to ensure that intervals remain small under repeated iterations of S. To this end, we define the growth function.

$$G : [0, 1] \to [0, 1]$$

$$G(x) = \sup\{m(S(I)) \,:\, I \subset [0, 1] \text{ an interval}\}$$

Above m represents the Lebesgue measure. We can note that since S is bounded and both piecewise monotonic and continuous, it follows that

$$\lim_{x \to 0} G(x) = 0\,.$$

We will use $G^n(x)$ to denote the n'th composition of G with itself, applied to x.

Definition 3.5.1. *We say that a collection of numbers $\{\epsilon_i\}_{i=0}^{\overline{n}-1}$ and intervals*

$$\left\{\{J_{i,j}\}_{j=1}^{(b_i - a_i)/\epsilon_i}\right\}_{i=0}^{\overline{n}-1}$$

give a ***basis of decomposition*** *if they fulfill the following conditions.*

1. *For each i, $G^{2N}(2\epsilon_i)$ is (j, N) admissible for all j.*
2. *For each i, $0 < G^{2N}(2\epsilon_i) < l$.*
3. *For each i and j, it holds that $J_{i,j} = [a_i + (j-1)\epsilon_i, a_i + j\epsilon]$.*
4. *For each i, $\frac{b_i - a_i}{\epsilon_i} \in \mathbb{N}$.*

It is clear fom the preceding lemmas that a basis of composition must exist.

Proposition 3.6. *Let $\{\epsilon_i\}$ and $\{J_{i,j}\}$ give a basis of decomposition. Choose some I to be an interval of length at most ϵ_i entirely contained in an I_i. Then I has a decomposition $(T_{2\overline{n}}, T_{[0,1]}, I_{e,E}, c_{e,E,i,j})$.*

Note that since (i, j) forms a finite set, we may index over (i, j) instead of a single index as in Definition 2.3.7 for the sake of simplicity.

Proof. Let $I_{(0),(0)} = I$. If $I_{e,E}$ is defined, and $|E| = |e| \not\equiv -1 \mod N$, we define

$$I_{e(0,i),E(0,t)} = \left\{x : x \in I_i, r \cdot \mathcal{P}(1_{I_{e,E}})(x) \geq t\right\}\,,$$

$$I_{e(0,\overline{n}+i),E(0,t)} = \emptyset\,,$$

and

$$c_{e,E,i,j} = 0\,.$$

Note that if $I_{e,E}$ is an interval entirely contained in some I_i, then so is each $I_{e(0,i),E(0,t)}$ by Lemma 1.3. Further

$$\bigcup_i I_{e(0,i),E(0,t)}$$

form the level set J_t of $\lambda \cdot P(1_{e,E})$.

If $I_{e,E}$ is defined and $|E| = |e| \equiv -1 \mod N$, we define

$$J_{e(0,i),E(0,t)} = \left\{x : x \in I_i, r \cdot \mathcal{P}(1_{I_{e,E}})(x) \geq t\right\}\,.$$

Note that if $I_{e,E}$ is an interval entirely contained in some I_j, then $J_{e(0,i),E(0,t)} = [a_{i,t}, b_{i,t}]$ is an interval entirely contained within I_i. Then there exist some $n_{i,t}$ minimal and $m_{i,t}$ maximal so that

$$a_i \leq a_{i,t} \leq a_i + n_{i,t} \cdot \epsilon_i \leq a_i + m_{i,t} \cdot \epsilon_i \leq b_{i,t} \leq b_i\,.$$

Recall that $I_i = [a_i, b_i]$. We now define $C_{i,j,t} = 1$ if $n_{i,t} < j \leq m_{i,t}$, and $C_{i,j,t} = 0$ otherwise. Finally, we can define our terms for the decomposition as

$$c_{e,E,i,j} = \int_0^1 C_{i,j,t}dt$$

$$I_{e(0,i),E(0,t)} = [a_{i,t}, a_i + n_{i,t} \cdot \epsilon_i]$$

and

$$I_{e(0,i+\overline{n}),E(0,t)} = [a_i + m_{i,t} \cdot \epsilon_i, b_{i,t}]\,.$$

Note that if $I_{e,E}$ is an interval entirely contained in some I_j, then both of our above intervals are well defined and contained in I_i. Further note that $c_{e,E,i,j} \leq 1$.

Then as so defined, notice that all of our intervals of form $I_{e(0,i),E(0,t)}$ or $I_{e(0,i+\overline{n}),E(0,t)}$ are entirely contained in I_i. Further, if $|e| \equiv 0 \mod N$, then the length of $I_{e,E}$ is at most $2\epsilon_i$ for $I_{e,E} \subset I_i$. Thus, for general e and E, if $|e| = |E| \equiv n \mod N$, with $0 \leq n \leq N$, then it follows that $I_{e,E}$ has length at most $G^n(\epsilon)$, where $\epsilon = \sup\{\epsilon_i\}$.

Since $|S_i'(x)| > r$, it follows that by our definitions so far,

$$r \cdot \mathcal{P}(1_{I_{e,E}}) = \int_0^1 \sum_{i=0}^{2\overline{n}-1} \left(1_{I_{e(0,i),E(0,t)}} + \sum_j C_{i,j,t} 1_{J_{i,j}}\right) dt\,.$$

Reordering and dividing both sides by r, we have that

$$\mathcal{P}(1_{I_{e,E}}) = \frac{1}{r}\left(\int \sum_{i=0}^{2\overline{n}-1} 1_{I_{e(0,i),E(0,t)}}(x)dt\right) + \frac{1}{r}\sum_{i,j}\int C_{i,j,t} \cdot 1_{J_{i,j}}dt\,.$$

It follows then that this construction fulfills Condition 1 of Definition 2.2.2. It clearly fulfills Condition 3.

In order to prove Condition 2, we proceed to show that few $I_{e,E}$ have non-zero measure. Notice that all of the $I_{e,E}$ that we have constructed are contained within some I_j and have length at most $\epsilon_i R^N$; further, $S(I_{e,E})$ intersects at most two distinct I_j. We define the splitting function

$$s(E) = \sum_{e \in T_k} 1_{I_{e,E} \neq \emptyset} .$$

Since $G^{2N}(2\epsilon_i)$ is (j, N) admissible, each $I_{e,E}$ splits at most once under N applications of $\mathcal{U}$. We then forcibly split them an additional time at the step where $|e| \equiv -1 \mod N$, and so it follows that if $F \subset E$ with $|F| = |E| - N$, and $|E| \equiv 0 \mod N$, then

$$s(E) \leq 4s(F) .$$

Then we may bound the quantity in Condition 2 of Definition 1.2.2 as follows. First note that we may rephrase it as

$$\sum_{e \in T_k} \frac{1}{r^{|e|}} \left(\int_{[0,1]^{|e|}} \left\| 1_{I_{e,E}} \right\|_{\mathrm{L}^1} dE \right) = \sum_{n=0}^{\infty} \frac{1}{r^n} \int_{|E|=n} \sum_{|e|=n} \left\| 1_{I_{e,E}} \right\|_{\mathrm{L}^1} dE .$$

Since $\left\| 1_{I_{e,E}} \right\|_{\mathrm{L}^1} \leq G^N(2\epsilon) \leq l$ for all e and E,

$$\sum_{n=0}^{\infty} \frac{1}{r^n} \int_{|E|=n} \sum_{|e|=n} \left\| 1_{I_{e,E}} \right\|_{\mathrm{L}^1} dE \leq \sum_{n=0}^{\infty} \frac{1}{r^n} \int_{|E|=n} s(E) \cdot l \cdot dE .$$

Using our work above,

$$\sum_{n=0}^{\infty} \frac{l}{r^n} \int_{|E|=n} s(E) dE \leq \sum_{n=0}^{\infty} \frac{1}{r^n} \int_{|E|=n} 4^{\frac{n}{N}+1} dE = 4 \sum_{n=0}^{\infty} \frac{4^{\frac{n}{N}}}{r^n} .$$

Recalling that $r^N > 5$, it is clear that

$$\frac{4^{\frac{n}{N}}}{r^n} < 1 ,$$

and so it follows that

$$\sum_{e \in T_k} \frac{1}{r^{|e|}} \left(\int_{[0,1]^{|e|}} \left\| 1_{I_{e,E}} \right\|_{\mathrm{L}^1} dE \right) < \infty$$

Thus all three conditions hold, and so we have constructed a decomposition. $\square$

4 Simplifying

We begin the section by introducing another notion of a decomposition.

Definition 4.0.1. *We say that a collection of non-negative functions $\{d_n\}$ in $L^1([0,1])$ and non-negative numbers $\{c_{n,j}\}$ in $\mathbb{R}$ a* ***simple decomposition*** *of an interval I by a finite collection of intervals $\mathcal{J}$ if it fulfills the following three conditions.*

1. $$\mathcal{P}(d_n) = \frac{1}{r}\left(d_{n+1} + \sum_j c_{n,j} \cdot 1_{J_j}\right)$$

2. $$\sum_{n=0}^{\infty} \frac{\|d_n\|_{L^1}}{r^{|e|}} < \infty$$

3. $d_0 = 1_I$

This allows us to work only on a countable collection of distribution functions, rather than our previous uncountable collection of intervals. The three conditions above are directly parallel to those in Definition 2.3.7, resulting in the following Lemma.

Lemma 4.1. *If I has a decomposition by $\mathcal{J}$, then it also has a simple decomposition by $\mathcal{J}$.*

Proof. Let $(T_k, T_{[0,1]}, I_{e,E}, c_{e,E,j})$ be a decomposition of I. Note that $I_{(0),(0)} = I$. Then we define

$$d_n(x) = \sum_{|e|=n} \int_{|E|=n} 1_{I_{e,E}}(x) dE$$

and

$$c_{n,j} = \sum_{|e|=n} \int_{|E|=n} c_{e,E,j} dE\,.$$

Then notice that by Definition 2.3.7,

$$Pd_n(x) = \sum_{|e|=n} \int_{|E|=n} \left(\frac{1}{r}\left(\int_{F\in D(E)} \sum_{f\in D(e)} 1_{I_{f,F}}(x) dF\right) + \frac{1}{r}\sum_j c_{e,E,j} \cdot 1_{J_j}\right)\,.$$

It is clear that the last terms, once separated, represent the $c_{n,j}$. We can also note that summing over $|e| = n$ and then over $f \in D(e)$ is equivalent to summing over $|f| = n+1$ directly. Similarly we reduce the integrals to an integral over $|F| = n+1$ and conclude that we do indeed have that

$$Pd_n = \frac{1}{r}\left(d_{n+1} + \sum_j c_{n,j} \cdot 1_{J_j}\right)\,.$$

This shows that condition 1 is fulfilled. Condition 2 is immediate from the second condition of Definition 2.3.7, as in Condition 3. □

Lemma 4.2. *If I has a simple decomposition by $\mathcal{J}$, $\{d_n\}$ and $\{c_{n,j}\}$, then*

$$\sum_j \left\|1_{J_j}\right\|_{L^1} \sum_{n=0}^{\infty} \frac{c_{n,j}}{r^{n+1}} = \left\|1_I\right\|_{L^1} .$$

Proof. By Condition 2 of Definition 3.0.1,

$$\lim_{n\to\infty} \frac{\|d_n\|_{\mathrm{L}^1}}{r^n} = 0 .$$

Then by the linearity of the L^1 norm for non-negative functions and that P preserves the L^1 norm of non-negative functions, notice that

$$\|1_I\|_{\mathrm{L}^1} = \|P^n d_0\|_{\mathrm{L}^1} = \sum_j \left\|1_{J_j}\right\|_{\mathrm{L}^1} \sum_{m=0}^{n-1} \frac{c_{m,j}}{r^{n+1}} + \frac{\|d_n\|_{\mathrm{L}^1}}{r^n} .$$

Taking n to infinity concludes the proof. □

Currently in order to show convergence we would have to work across $|\mathcal{J}|$ sequences simultaneously. What follows in this section will reduce us to being able to independently work on each sequence, only combining them in the final steps. To do this, we introduce the following definitions.

Definition 4.2.1. *We call a finite collection of intervals $\mathcal{J}$ a **general basis of decomposition for** ϵ if $\epsilon > 0$ and every interval of length at most ϵ has a simple decomposition by $\mathcal{J}$. We further require that each $J_j \in \mathcal{J}$ has a simple decomposition by $\mathcal{J}$.*

Definition 4.2.2. *We call a collection of intervals $\mathcal{J}$ a **diagonal basis of decomposition** if it is a general basis of decomposition and for each $J_j \in \mathcal{J}$ there exists a simple decomposition of J_j by $\mathcal{J}$, $\{d_n\}$ and $\{c_{n,i}\}$ such that $c_{n,i} > 0$ only if $i = j$.*

It could be difficult to completely disentangle a general basis of decomposition. Instead we knit any "entangled" sequences into the other sequences; this leaves us with one less interval in $\mathcal{J}$. Eventually we will either have only a single sequence remaining, or the sequences would form a diagonal basis of decomposition.

Proposition 4.3 (Decomposition Knitting Proposition)**.** *Let $\mathcal{J}$ be a general basis of decomposition for $\epsilon > 0$. If $\mathcal{J}$ is not a diagonal basis of decomposition, then there exists a $\mathcal{J}' \subset \mathcal{J}$ that is also a general basis of decomposition for ϵ.*

Proof. By hypothesis there is some $\jmath$ so that the simple decomposition of $J_\jmath \in \mathcal{J}$ by $\mathcal{J}$ is $(D_n, C_{n,j})$ with some some $j \neq \jmath$ and $n \in \mathbb{Z}_{\geq 0}$ such that $C_{n,j} > 0$. We will aim to show that $\mathcal{J} \setminus \{J_\jmath\}$ is a general decomposition.

Let I either be an interval of length at most ϵ or an element of $\mathcal{J}$ such that $I \neq J_{\jmath}$. Then I has a simple decomposition by $\mathcal{J}$, $\{d_n\}$ and $\{c_{n,j}\}$. We will now inductively define a sequence of simple decompositions of I, beginning with $d_n^0 = d_n$ and $c_{n,j}^0 = c_{n,j}$. If d_n^m and $c_{n,j}^m$ are already defined, then we let

$$d_n^{m+1} = d_n^m + \sum_{k=0}^{n-1} c_{k,\jmath}^m \cdot D_{n-1-k}$$

and

$$c_{n,j}^{m+1} = 1_{j \neq \jmath} \cdot c_{n,j}^m + \sum_{k=0}^{n-1} c_{k,\jmath}^m \cdot C_{n-1-k,j} \,.$$

This may be an initially intimidating condition. The idea is that whenever $c_{n,\jmath}^m > 0$ we attach an entire copy of D_n there. This allows us to take $c_{n,\jmath}^{m+1} = 0$. Because these attachments are displaced, summing across all of the attached sequences gives the second term in the definition of d_n^{m+1}; we recommend that the reader convince themselves of this fact before proceeding, as it is at the heart of this proof. What follows are largely formalisms to ensure that our construction does indeed behave well.

We will now show that d_n^{m+1} and $c_{n,j}^{m+1}$ form a simple decomposition of I if d_n^m and $c_{n,j}^m$ did. For the first condition, note that

$$P(d_n^{m+1}) = \frac{1}{r}\left(d_{n+1}^m + \sum_j c_{n,j}^m 1_{J_j} + \sum_{k=0}^{n-1} c_{k,\jmath}^m \cdot \left(D_{n-k} + \sum_j C_{n-1-k,j} 1_{J_j}\right)\right) .$$

Noticing that $c_{n,\jmath}^m 1_{J_\jmath} = c_{(n+1)-1,\jmath} \cdot D_0$ and that $n-k = (n+1)-1-k$, it is clear that reordering terms does indeed give us

$$P(d_n^{m+1}) = \frac{1}{r}\left(d_{n+1}^{m+1} + \sum_j c_{n,j}^m 1_{J_j}\right) .$$

Next, we aim to bound

$$\sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \cdot c_{n,\jmath}^m$$

in m. Notice that

$$\sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \cdot c_{n,\jmath}^{m+1} = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \left(1_{\jmath \neq \jmath} \cdot c_{n,\jmath}^m + \sum_{k=0}^{n-1} c_{k,\jmath}^m \cdot C_{n-1-k,\jmath}\right) .$$

Since here, $j = \jmath$, we may drop the leading terms. We then exchange the order of summation to get

$$\sum_{k=0}^{\infty} c_{k,\jmath}^m \sum_{n=k+1}^{\infty} \frac{1}{r^{n+1}} \cdot C_{n-1-k,\jmath} = \sum_{k=0}^{\infty} \frac{1}{r^{k+1}} \cdot c_{k,\jmath}^m \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \cdot C_{n,\jmath} \,.$$

By hypothesis and Lemma 3.2, we may define

$$0 < \delta = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \cdot C_{n,\mathrm{J}} < 1$$

and so,

$$\sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \cdot c_{n,\mathrm{J}}^{m+1} = \delta \sum_{n} \frac{1}{r^{n+1}} c_{n,\mathrm{J}}^{m}$$

Next, we consider that

$$\sum_{n=0}^{\infty} \frac{\left\|d_n^{m+1}\right\|_{\mathrm{L}^1}}{r^n} = \sum_{n=0}^{\infty} \frac{1}{r^n} \left(\|d_n^m\|_{\mathrm{L}^1} + \sum_{k=0}^{n-1} c_{k,\mathrm{J}}^m \cdot \|D_{n-1-k}\|_{\mathrm{L}^1} \right) .$$

We may reorder the sums, so that we instead examine the quantity

$$\left(\sum_{n=0}^{\infty} \frac{\|d_n^m\|_{\mathrm{L}^1}}{r^n} \right) + \sum_{k=0}^{\infty} c_{k,\mathrm{J}}^m \sum_{n \geq k+1} \frac{1}{r^n} D_{n-1-k} .$$

If we let

$$L = \sum_{n=0}^{\infty} \frac{\|D_n\|_{\mathrm{L}^1}}{r^n} < \infty$$

then we may rewrite the sum once more as

$$\sum_{n} \frac{\|d_n^m\|_{\mathrm{L}^1}}{r^n} + L \sum_{k=0}^{\infty} \frac{c_{k,\mathrm{J}}^m}{r^{k+1}} = \sum_{n} \frac{\|d_n^m\|_{\mathrm{L}^1}}{r^n} + L\delta^m \sum_{k=0}^{\infty} \frac{c_{k,\mathrm{J}}^0}{r^{k+1}} .$$

Since this has an exponential decay δ for the amount added, it follows that these sums may be bounded above by a fixed constant C independent of m.

What we have shown then is that we can create a sequence of decomposition whose reliance on J_{J} tends to zero. We now define the limiting decomposition and show that it has the appropriate properties.

We conclude the proof by firstly noting that for a fixed m, if n is the infimal number such that $c_{n,\mathrm{J}}^m > 0$, then it follows from our definitions that for all $p \leq n$, $c_{p,\mathrm{J}}^{m+1} = 0$. From this it is clear that if $m > n$, then $c_{n,\mathrm{J}}^m = 0$, and thus that $d_n^m = d_n^{m+1}$. We define

$$d_n^{\infty} = d_n^{n+2}$$
$$c_{n,j}^{\infty} = c_{n,j}^{n+2} .$$

We claim that this forms a decomposition of I. By our note above,

$$Pd_n^{\infty} = \frac{1}{r} \left(d_{n+1}^{n+2} + \sum_j c_{n,j}^{n+2} 1_{J_j} \right) = \frac{1}{r} \left(d_{n+1}^{n+3} + \sum_j c_{n,j}^{\infty} 1_{J_j} \right) ,$$

which is equal to what we need for Condition 1 to hold. Condition 2 holds since d_n^m monotonically increase, but their sum, normalized by $\frac{1}{r^n}$, is uniformly bounded above by C. Finally, Condition 3 holds immediately. Then we notice that $c^{\infty}_{n,\mathrm{J}} = 0$ for all n, and thus I was decomposed by $\mathcal{J} \setminus \{J_{\mathrm{J}}\}$.

$\square$

Corollary 4.3.1. *If $\mathcal{J}$ is a general basis of decomposition, then there exists some $\mathcal{K} \subset \mathcal{J}$ that is a diagonal basis of decomposition.*

5 Convergence

Let I be a fixed element of a diagonal basis of decomposition $\mathcal{J}$. Then we fix its decomposition d_n and $c_{n,j}$; note that we may reduce $c_{n,j}$ to a single sequence c_n, as $\mathcal{J}$ is diagonal. We then define the invariant distribution

$$d = \alpha \sum_{n=0}^{\infty} \frac{1}{r^n} d_n \,,$$

where $\alpha > 0$ is a constant chosen so that $\|d\|_{\mathrm{L}^1} = \|1_I\|_{\mathrm{L}^1}$. By Lemma 3.2, it is immediate that $Pd = d$. From here we have a natural definition for what we will call the general representation of $P^m(1_I - d)$.

Definition 5.0.1. *For a fixed I an element of a diagonal basis of decomposition, we define its general representation $\{a^m_n\}$ of $P^m(1_I - d)$ inductively as*

$$a^0_0 = 1 - \alpha$$

$$a^0_n = \frac{-\alpha}{r^n} \text{ for } n > 0$$

$$a^{m+1}_0 = \frac{1}{r} \sum_{n=0}^{\infty} c_n \cdot a^m_n$$

$$a^{m+1}_n = \frac{a^m_n}{r} \text{ for } n > 0\,.$$

Lemma 5.1. *The following equality holds for all m.*

$$P^m(1_I - d) = \sum_{n=0}^{\infty} a^m_n \cdot d_n$$

Proof. This is clearly true for $m = 0$. But by the linearity of P and that d_n and c_n form a simple decomposition, it must hold for all m. $\square$

This tells us that the a^m_n are indeed representing $P^m(1_I - d)$, so we would like to show that they have useful properties. Firstly we can show that they decay exponentially in n.

Lemma 5.2. *For all n and m, the following inequality holds.*

$$a^m_n \leq \frac{2}{r^n}$$

Proof. Since $\|Pf\|_{\mathrm{L}^1} \leq \|f\|_{\mathrm{L}^1}$ for all $f \in L^1([0,1])$, it follows that

$$\sum_n |a_n^m| \, \|d_n\|_{\mathrm{L}^1} \leq 2 \, \|d_0\|_{\mathrm{L}^1} \ ,$$

and so necessarily $|a_0^m| \leq 2$. Notice that

$$|a_n^m| = \left| \frac{1}{r^m} a_{n-m}^0 \right| = \left| \frac{1}{r^m} \cdot \frac{-\alpha}{r^{n-m}} \right| \leq \frac{2}{r^n} \ \text{ if } n > m \text{ and}$$

$$|a_n^m| = \left| \frac{1}{r^n} a_0^{m-n} \right| \leq \frac{2}{r^n} \ \text{otherwise.}$$

□

We would like a reasonable way to measure the size of the a_n^m in m. To do so, we define the weight function, which is a tailored analogue of ℓ^1.

Definition 5.2.1. *The **weight at step** **m**, $W(m)$ is defined as*

$$W(m) = \sum_{n=0}^{\infty} |a_n^m| \cdot \|d_n\|_{L^1} \ .$$

It is an important remark that

$$W(m) \geq \|P^m(1_I - d)\|_{\mathrm{L}^1} \ ,$$

and so it suffices to show that $W(m)$ goes to 0. Since $\|P^m(1_I - d)\|_{\mathrm{L}^1}$ is monotonic decreasing in m, we can show the following analogue for W.

Lemma 5.3. *$W(m)$ decreases monotonically in m.*

Proof. Notice that

$$W(m+1) = \sum_{n=0}^{\infty} \left| a_n^{m+1} \right| \cdot \|d_n\|_{\mathrm{L}^1}$$

Applying our definitions for a_0^m and a_n^m for $n > 0$ separately, we can rephrase the above as

$$\frac{\|d_0\|_{\mathrm{L}^1}}{r} \left| \sum_{n=0}^{\infty} c_n \cdot a_n^m \right| + \frac{1}{r} \sum_{n=0}^{\infty} a_{n-1}^m \, \|d_{n+1}\|_{\mathrm{L}^1}$$

and so by the triangle inequality,

$$W(m+1) \leq \frac{\|d_0\|_{\mathrm{L}^1}}{r} \sum_{n=0}^{\infty} c_n \cdot |a_n^m| + \frac{1}{r} \sum_{n=0}^{\infty} a_n^m \, \|d_{n+1}\|_{\mathrm{L}^1} \ .$$

Because d_n and c_n form a simple representation, it follows that

$$\frac{1}{r} \left(\|d_{n+1}\|_{\mathrm{L}^1} + c_n \, \|d_0\|_{\mathrm{L}^1} \right) = \|d_n\|_{\mathrm{L}^1} \ ,$$

and so combining these terms we arrive at our result. □

The monotonicity of the weight means that we need only show that we can obtain a weight decrement. We begin by showing in Lemma 4.6 that if the weight stays above some value ϵ, then a_0^m has obtains "large" positive and negative values. Lemma 4.7 says that high oscillation in a_0^m at certain times means that a weight decrement will occurr. Lemma 4.8 combines the two to show that a weight decrement can always be obtained.

Lemma 5.4. *For all $\epsilon > 0$ there exist a $\delta > 0$ and a number $\overline{m}$ such that for all $m \in \mathbb{Z}_{\geq 0}$ one of the following holds.*

1. *There exist some $0 \leq m_1, m_2 \leq \overline{m}$ such that $\delta < a_n^{m+m_1}$ and $-\delta > a_n^{m+m_2}$.*
2. *$W(m + \overline{m}) \leq \epsilon$.*

Proof. Recall that $\int f = \int Pf$. Then it follows that

$$0 = P^m \int (1_I - d) = \int \left(\sum_{n=0}^{\infty} a_n^m d_n \right)$$

and so

$$\sum_{n=0}^{\infty} |a_n^m|^+ \|d_n\|_{\mathrm{L}^1} = \sum_{n=0}^{\infty} |a_n^m|^- \|d_n\|_{\mathrm{L}^1} \ .$$

Above we use $|\cdot|^+$ and $|\cdot|^-$ to denote the positive and negative parts of the sequences respectively.

Now fix an $\epsilon > 0$ and choose an $\overline{m}$ sufficiently large that

$$\sum_{n=\overline{m}}^{\infty} \frac{2}{r^n} < \frac{\epsilon}{4} \, .$$

Choose $\delta > 0$ sufficiently small that $\overline{m}\delta < \frac{\epsilon}{4}$. Notice that if for each $0 \leq m_1 \leq \overline{m}$ we have that $a_n^{m+m_1} \leq \delta$, then for each $n \leq \overline{m}$,

$$a_n^{m+\overline{m}} = \frac{a_0^{m+\overline{m}-n}}{r^n} \leq \delta \, .$$

It follows then that

$$\sum_{n=0}^{\infty} \left| a_n^{m+\overline{m}} \right|^+ \|d_n\|_{\mathrm{L}^1} \leq \delta \overline{m} + \sum_{n=\overline{m}}^{\infty} \frac{2}{r^n} < \frac{\epsilon}{2} \, .$$

Thus $W(m + \overline{m}) \leq \epsilon$. The argument follows almost identically for m_2. $\square$

Lemma 5.5. *Suppose that there exist some $p, q \in \mathbb{Z}_{\geq 0}$ such that $a_0^p > 0$ and $c_q > 0$. If*

$$a_0^{p+q+1} < \frac{c_q a_0^p}{2r^{q+1}}$$

then

$$W(p + q + 1) \leq W(p) - \frac{c_q a_0^p \|d_0\|_{L^1}}{2r^{q+1}}$$

Proof. Notice that

$$W(p+q+1) + \frac{1}{r}\sum_{n=0}^{\infty} c_n \left|a_n^{p+q}\right| \|d_0\|_{\mathrm{L}^1} - \left|a_0^{p+q+1}\right| \|d_0\|_{\mathrm{L}^1}$$

is equal to

$$\frac{1}{r}\sum_{n=0}^{\infty} c_n \left|a_n^{p+q}\right| \|d_0\|_{\mathrm{L}^1} + \frac{1}{r}\sum_{n=0}^{\infty} \left|a_n^{p+q}\right| \|d_{n+1}\|_{\mathrm{L}^1} = \sum_{n=0}^{\infty} \left|a_n^{p+q}\right| \|d_n\|_{\mathrm{L}^1} = W(p)\,.$$

Then it follows that

$$W(p+q+1) - W(p) = \left|a_0^{p+q+1}\right| \|d_0\|_{\mathrm{L}^1} - \frac{1}{r}\sum_{n=0}^{\infty} \left|a_n^{p+q}\right| \|d_0\|_{\mathrm{L}^1}\ .$$

Notice that we must have that

$$a_q^{p+q} = \frac{a_0^p}{r^q}$$

and that

$$\frac{c_q a_0^p}{2r^{q+1}} > a_0 = \frac{1}{r}\sum_{n=0}^{\infty} c_n a_n^{p+q}\,.$$

Then the proof reduces, up to reordering so that $x_0 = c_q a_q^{p+q}$, to showing that for a sequence of real numbers x_n with $0 < x_0$,

$$\sum_{n=0}^{\infty} x_n < \frac{x_0}{2} \implies \left|\sum_{n=0}^{\infty} x_n\right| - \sum_{n=0}^{\infty} |x_n| \le -\left|\frac{x_0}{2}\right|\,.$$

We denote $S^+ = \sum_{n=0}^{\infty} |x_n|^+$ and $S^- = \sum_{n=0}^{\infty} |x_n|^-$ to be the positive and negative parts of the sum respectively. Since $S^+ \ge x_0$ it follows that $S^- \ge x_0/2$. Notice that

$$\left|\sum_{n=0}^{\infty} x_n\right| - \sum_{n=0}^{\infty} |x_n| \le -\left|\frac{x_0}{2}\right| = -2\inf\left(S+, S-\right) \le -\frac{|x_0|}{2}\,.$$

□

Proposition 5.6. *If* $\gcd\{n+1 : c_n > 0\} = 1$*, then for every* $L > 0$ *there exist some* $\alpha > 0$ *and* $\overline{n} \in \mathbb{N}$ *such that for every* p *with* $W(p) > L$*,* $W(p+\overline{n}) \le W(p) - \alpha$*.*

Proof. Fix $L > 0$ and p such that $W(p) > L$. Choose $\overline{m}$ as in Lemma 4.6 and $\epsilon = L/2$. Notice then that by Lemma 4.6, either $W(p+m) < L/2 < W(p) - L/2$, or Condition 1 of Lemma 4.6 holds. Let $\alpha < L/2$ and assume that Condition 1 holds, as if not we are done.

We claim that there exist some $Q \in \mathbb{N}$, $M \in \mathbb{N}$ and $1 \ge \lambda > 0$ so that for every m_1 and m_2 there exist two sequences $\{A_j\}_{j=1}^{h}$ and $\{B_j\}_{j=1}^{\ell}$ such that the following hold.

1. $h, \ell \leq Q$
2. For all $i \leq h$ and $j \leq \ell$, we have that A_i and B_j are at most M.
3. For all $i \leq h$ and $j \leq \ell$, each c_{A_i} and c_{B_j} are at least λ.
4. $$m_1 + \sum_{i=1}^{h}(A_i + 1) = m_2 + \sum_{i=1}^{\ell}(B_i + 1)\ .$$

This follows from $\gcd\{i+1 : c_i > 0\} = 1$ and that there are finitely many combinations $0 \leq m_1, m_2 \leq \overline{m}$.

We have the hypothesis that $a_0^{p+m_1} > \delta$. We now consider $a_0^{p+m_1+A_1+1}$. By Lemma 4.7, either

$$a_0^{p+m_1+A_1+1} \geq \frac{c_{A_1} a_0^{p+m_1}}{2r^{A_1+1}} \geq \frac{\lambda\delta}{2r^{M+1}}$$

or

$$W(p+m_1+A_1+1) \leq W(p+m_1) - \frac{c_{A_1} a_0^{p+m_1} \|d_0\|_{\mathrm{L}^1}}{2r^{A_1+1}} \leq W(p+m_1) - \frac{\lambda\delta \|d_0\|_{\mathrm{L}^1}}{2r^{M+1}}\ .$$

If the second case holds, then for α chosen sufficiently small (as δ, λ, r, and M are all independent of L) our result holds. Then assuming that case 1 holds, we may repeat the argument so that either

$$a_0^{p+m_1+A_1+1+A_2+1} \geq \frac{c_{A_1} c_{A_2} a_0^{p+m_1}}{2^2 \cdot r^{(A_1+1+A_2+1)}} \geq \frac{\lambda^2\delta}{2^2 \cdot r^{2M+2}}$$

or

$$W(p+m_1+A_1+1+A_2+1) \leq W(p+m_1) - \frac{\lambda^2\delta \|d_0\|_{\mathrm{L}^1}}{2^2 \cdot r^{2M+2}}\ .$$

We repeat this argument h times so that in the end we conclude that either

$$a_0^{\left(p+m_1+\sum_{i=1}^{h}(A_i+1)\right)} \geq \frac{\lambda^h\delta}{2^h r^{hM+h}} \geq \frac{\lambda^Q\delta}{2^Q r^{Q(M+1)}}$$

or

$$W\left(p + m_1 + \sum_{i=1}^{h}(A_i+1)\right) \leq W(p+m_1) - \frac{\lambda^Q\delta \|d_0\|_{\mathrm{L}^1}}{2^Q r^{M(Q+1)}}\ .$$

Repeating a similar argument on the B_i we may also conclude that either

$$a_0^{\left(p+m_1+\sum_{i=1}^{h}(B_i+1)\right)} \leq \frac{-\lambda^h\delta}{2^h r^{hM+h}} \leq \frac{-\lambda^Q\delta}{2^Q r^{Q(M+1)}}$$

or

$$W\left(p + m_2 + \sum_{i=1}^{h}(B_i+1)\right) \leq W(p+m_1) - \frac{\lambda^Q\delta \|d_0\|_{\mathrm{L}^1}}{2^Q r^{Q(M+1)}}\ .$$

Recalling that

$$p + m_1 + \sum_{i=1}^{h} (A_i + 1) = p + m_2 + \sum_{i=1}^{h} (B_i + 1)$$

it is impossible that

$$a_0^{\left(p+m_1+\sum_{i=1}^{h}(B_i+1)\right)} < 0 < a_0^{\left(p+m_1+\sum_{i=1}^{h}(A_i+1)\right)} .$$

Thus we must have a weight decrement of size

$$\alpha < \frac{\lambda^Q \delta \left\| d_0 \right\|_{\mathrm{L}^1}}{2^Q r^{Q(M+1)}} .$$

Applying the monotonicity of $W(m)$ from Lemma 4.5 we conclude the proof. □

This is very nearly what we need. However, it has the problematic condition that $\gcd \{n + 1 : c_n > 0\} = 1$. We now present an easy lemma that can take a higher gcd sequence and turn it into a sequence with gcd equal to 1. We first note that if S fulfills the conditions listed in section 1, then so to does S^k for $|S'| \geq r^K > 1$, and all of our results apply to P^k as well.

Lemma 5.7. *Suppose that an interval J has a simple decomposition with respect to itself, that is, $\{J\}$. We denote this decomposition d_n and c_n. If $K = \gcd \{n + 1 : c_n > 0\} = 1$ then d_{Kn} and $c_{K(n+1)-1}$ form a simple decomposition of J with respect to $\{J\}$ for the operator P^K.*

Proof. Notice that

$$P^K(d_{Kn}) = \frac{1}{r^K} \left(d_{Kn+K} + \sum_{m=0}^{K-1} c_{Kn+m} d_{K-m-1} \right) .$$

By hypothesis, each $c_{Kn+m} = 0$ except for c_{Kn+K-1}, and so

$$P^K(d_{Kn}) = \frac{1}{r^k} \left(d_{Kn+K} + c_{Kn+K-1} d_0 \right) .$$

From this it is clear that d_{Kn} and $c_{K(n+1)-1}$ fulfill Condition 1 of Definition 3.0.1. Condition 2 follows by the dominated series test and Condition 3 is trivial. □

Corollary 5.7.1. *Suppose that an interval J has a simple decomposition with respect to itself, $\{J\}$. We denote this decomposition d_n and c_n. If $K = \gcd \{i + 1 : c_i > 0\} = 1$ then $P^K(1_J)$ converges to some f in the L^1 sense.*

We at last prove Theorem 1.2.

Proof of Theorem 1.2. □

By Proposition 2.6 a general basis of decomposition $\mathcal{J}$ exists. By Corollary 3.3.1 there is some diagonal basis of decomposition for $\epsilon > 0$ $\mathcal{K} \subset \mathcal{J}$. Fix an interval I of length at most ϵ.

Then let d_n and $c_{n,k}$ be the general decomposition of I by $\mathcal{K}$ such that

$$P(d_n) = \frac{1}{r}\left(d_n + \sum_k c_{n,k} 1_{J_k}\right)$$

where each $J_k \in \mathcal{K}$.

By Corollary 4.9.1, for each $J_k \in \mathcal{K}$ there exists some integer K_k such that $P^{K_k} 1_{J_k}$ converges to some f_k in the L^1 sense. Let K be the least common multiple of the K_k. Then notice that

$$P^{nK}(d_0) = \frac{1}{r^{nK}} d_{nK} + \sum_k \sum_{m=0}^{nK-1} \frac{c_{m,k}}{r^{m+1}} \cdot P^{nK-m-1} 1_{J_k} \, .$$

Since $\frac{1}{r^{nK}} d_{nK}$ goes to 0 in n, and

$$\left\| \sum_k \sum_{m=0}^{nK-1} \frac{c_{m,k}}{r^{m+1}} \cdot P^{nK-m-1} 1_{J_k} \right\|_{\mathrm{L}^1} \leq \left\| P^{nK}(d_0) \right\|_{\mathrm{L}^1} = \epsilon$$

it suffices to show by the Dominated Convergence Test that each

$$\frac{c_{m,k}}{r^{m+1}} \cdot P^{nK-m-1} 1_{J_k}$$

converges in L^1. Let $m' \equiv -m-1 \mod K$. Then it follows by Corollary 4.9.1 that

$$\lim_{n\to\infty} \frac{c_{m,k}}{r^{m+1}} \cdot P^{nK-m-1} 1_{J_k} = \frac{c_{m,k}}{r^{m+1}} P^{m'} \left(\lim_{n\to\infty} P^{nK} 1_{J_k}\right) = \frac{c_{m,k}}{r^{m+1}} P^{m'} (f_k) \, .$$

Thus, $P^{nK}(1_I)$ converges in L^1 to a linear combination of the set

$$\{\{P^m f_k\}_k\}_{m=1}^{K} \, ,$$

which concludes our proof.