A Constructive Proof for the Asymptotic Periodicity of Mappings with Piecewise Monotonic Derivatives

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Abstract

In their 1984 paper, Lasota, Li, and Yorke presented an argument that if $S : [0, 1] \to [0, 1]$ is piecewise C^2 with $\inf |S'| > 1$, then its associated Frobenius-Perron operator is asymptotically periodic. These results have been generalized in later works, primarily with functional-analytic methods using bounded variation. In this paper we present a novel method to prove a past result using constructive techiques and the Spectral Decomposition Theorem.

1 Introduction

If (X, μ) is a measure space, μ is nonsingular, and $S : X \to X$ is a measurable transformation, then a fundamental question is how its iterates $S^n : X \to X$, i.e. the dynamical system generated by S, behave. In studying such dynamical systems, often the behavior of densities of points under repeated applications of an operator S is examined, rather than the behavior of individual points. By a density, we mean a measurable, non-negative function $f : X \to \mathbb{R}$ such that $\int_X f d\mu = 1$. We define the action of the transformation S on f as the (unique) associated Frobenius-Perron operator $P : L^1(x) \to L^1(x)$, which satisfies that for every $A \in X$,

$$\int_A P f d\mu = \int_{S^{-1}(A)} f d\mu \,.$$

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It follows that if $\mathcal{X}$ is a random variable with density function f, then $S(\mathcal{X})$ is a random variable with density function Pf. From this framing, the question of whether $S^n(\mathcal{X})$ will converge in distribution arises, and it suffices to examine the convergence of the density functions $P^n f$ in L^1. If every density function f converges, we say that P is *asymptotically stable*. Since examining the behavior of the random variables $S^n(\mathcal{X})$ is sometimes more tractable than working with the individual orbits of $S^n(x)$, it can be advantageous to explore this functional analytic point of view more deeply.

In this paper we focus on dynamical systems on the unit interval $X = [0,1]$ given by a function $S : X \to X$ that is piecewise smooth and satisfies $|S'(x)| \geq r > 1$ for all $x \in [0,1]$. The stochastic properties of such dynamical systems have been extensively studied, going back to Rényi [15], who established the existence of an invariant distribution f_*, satisfying $Pf_* = f_*$, for the functions $S(x) = rx \mod 1$. The existence of an invariant density f_* is crucial in understanding the dynamical system generated by S, as the corresponding distribution μ_*, defined by $\mu_*(A) := \int_A f_* d\mu$, is invariant under S, and thus may be examined under ergodic theoretic methods.

This line of inquiry was explored further by Lasota and Yorke [10], who proved the existence of invariant distributions for certain functions S that are piecewise C^2. These results were further developed in [12], [1], [19], and [18] where ergodic properties of similar dynamical systems have been examined. Related questions are still studied, but are beyond the scope of this paper. The approaches used in these papers are also useful for studying the convergence of Cesaro means $\frac{1}{N}\sum_{n\leq N} P^n f$ and, to a lesser extent, for the convergence of the sequence $P^n f$. However, while an invariant density may exist, $P^n f$ need not converge in L^1, and so the methods struggle to generalize to these questions. For example, for any piecewise C^2 function S, let

$$T = \begin{cases} \frac{S(2x)+1}{2} & : 0 \leq x < \frac{1}{2} \\ \frac{S(2x-1)}{2} & : \frac{1}{2} \leq x < 1 \end{cases}.$$

Let P be the Frobenius-Perron operator associated to T. It is clear that if a function f has support in $[0, 0.5)$, then Pf must have support in $[0.5, 1]$ and vice versa; hence $P^n f$ does not converge in L^1.

To deal with such phenomena Hofbauer and Keller in [5] introduced the concept of *asymptotic periodicity*, meaning that there exists some $K \in \mathbb{N}$ so that $P^{nK} f$ converges in L^1 for all distributions f. Lasota and Yorke obtained an initial result in [9]. This was later strengthened by Keller [7] to show that if a function S is piecewise C^1; fulfills $|S'| > r > 1$ for some r; and for each C^1 piece $S|_{I_i}$, $1/S|'_{I_i}$ has bounded variation, then S has an asymptotically periodic Frobenius-Perron operator. Keller's work used the ergodic theorem of Ionescu Tulcea and Marinescu [17]. These results were further generalized by Rychlik [16]; Liverani [13]; Jabłonski and Góra [14]; Bugiel [2]; Góra [3]; Góra, Li, and Boyarsky [4]; and Islam [6]. The newer results focus on functions fulfilling certain bounds on variation and oscillation.

In this paper we present a novel, constructive argument that sets up for the application of the Spectral Decomposition Theorem by Lasota and Yorke [9]. Below, we summarize the important definitions, as well as our results.

Definition 1.1. *We say that a functions $S : [0,1] \to [0,1]$ is a **piecewise r-dilation** if it fulfills the following conditions. Let the set*

$$B = \{x : S'(x) \textit{does not exist}\} .$$

1. *There is a finite collection of disjoint open intervals over an index set $\mathcal{I}$, $\{I_i\}_{i\in\mathcal{I}}$, whose closures cover $[0,1]$. For each $i \in \mathcal{I}$, the restriction of S to $I_i \backslash B$ has a monotonic derivative. Further the restriction of S to I_i is continuous and injective.*
2. *For every $x \in [0,1]\backslash B$, $|S'(x)| \geq r > 1$.*
3. *The sets B and $S(B)$ have Lebesgue measure 0.*

Theorem 1.2. *If S is a piecewise r-dilation for $r > 1$, then the associated Frobenius-Perron operator is asymptotically periodic.*

Above is the main theorem that we prove in this paper. Rather than require a condition on the variation of the first derivative, we use a monotonicity assumption. This condition is implied by the hypotheses of Lasota and Yorke in [9], and implies the hypotheses of Keller in [7]. To our knowledge, all work thus far has focused on manipulating variation and oscillation, generally showing that the Frobenius-Perron operator maps functions into a space where the variation of all of the functions is bounded by some C, which is sufficient to apply the Spectral Decomposition Theorem of Lasota and Yorke [9] or a similar theorem. Here, we require the monotonicity of S' for clarity and ease of comparison with other theorems, though it is the following equivalent condition that we truly need.

Remark. *A function S is a piecewise r-dilation if and only if it fulfills the following conditions.*

Let the set

$$B = \{x : S'(x) \textit{does not exist}\} .$$

1. *There is a finite collection of disjoint open intervals over an index set $\mathcal{I}$, $\{I_i\}_{i\in\mathcal{I}}$, whose closures cover $[0,1]$. For each $i \in \mathcal{I}$ the restriction of S to I_i is continuous and injective.*
2. *For every $x \in [0,1]\backslash B$, $|S'(x)| \geq r > 1$.*
3. *The sets B and $S(B)$ have Lebesgue measure 0.*
4. *Let P be the Frobenius-Perron operator associated to S, and let $I \subseteq I_i$ for some I. Then if L_t is the set of all points where $P1_I > t$, then L_t can be expressed as an interval.*

Definition 1.3. *We call a collection of non-negative functions $\{d_n\}$ in $L^1([0,1])$ and non-negative numbers $\{c_{n,j}\}$ in $\mathbb{R}$ a* ***simple decomposition*** *of an interval I if $d_0 = 1_I$,*

1.
$$P(d_n) = \frac{1}{r}\left(d_{n+1} + \sum_{j\in\mathcal{I}} c_{n,j}\cdot 1_{I_j}\right)$$
for $n \geq 0$, and

2.
$$\sum_{n=0}^{\infty} \frac{\|d_n\|_{L^1}}{r^n} < \infty\,.$$

The proof of Theorem 1.2 is based on the following proposition.

Proposition 1.4. *Let S be a piecewise r-dilation for $r > 2$ over a finite collection $\{I_i\}_{i\in\mathcal{I}}$. If I is an interval such that $I \subseteq I_i$ for some $i \in \mathcal{I}$, then I has a simple decomposition with each $\|d_n\|_{L^1} \leq 2^n$.*

We will discuss this section of the proof more in Section 3, but the construction for Proposition 1.4 relies on the fact that after each application of P to a distribution f, we may then examine applying P to the level sets of Pf, rather than Pf itself. We then apply a trimming process, in which we are able to separate any part of the interval that entirely contains an I_i, resulting in the $c_{n,j}\cdot 1_{I_j}$ terms. This splits the original interval into two smaller pieces, each of which fulfill the conditions in Lemma 1, and so we may apply P and repeat the process. We compose d_n by summing over all of the pieces that result after n iterations, and integrating across all of the level sets. Since $0 \leq t \leq 1$, and each piece splits into at most two pieces, we have that $\|d_n\|_{\mathrm{L}^1}\, 1 \leq 2^n$. Since S is a piecewise dilation of $r > 2$, it follows that a sum over $(2/r)^n$ converges, giving condition 2 of Definition 1.3.

It may be helpful to the reader to see these decompositions in action, and so for the remainder of the introduction, we will look at a setting originally examined by Rényi [15]. In this setting, we may construct the simple decomposition directly, though the process is a simplified version of what we will do in Section 3. Let $S = rx \mod 1$, or more formally,

$$S(x) = \begin{cases} rx - \lfloor rx \rfloor & : x \in [0,1] \\ 0 & : x \notin [0,1] \end{cases}$$

and $r \approx 1.6$ satisfies $r(r-1) = 1$. In this setting, Rènyi proved asymptotic stability; while it is possible to do a general convergence argument that can also give asymptotic stability here, it is fairly long and tedious. Instead, we will use the Spectral Decomposition Theorem of Lasota and Yorke [9], though this restricts us to asymptotic periodicity without an additional argument. In this setting, we may explicitly state the associated Frobenius-Perron operator $P : L^1([0,1]) \to L^1([0,1])$ as

$$P(f)(x) = \frac{1_{[0,1]}(x)}{r} \sum_{z=0}^{1} f\left(\frac{x-z}{r}\right).$$

We discuss obtaining such explicit constructions of the Frobenius-Perron operator at the beginning of Section 3. As we are working with L^1 functions, our equalities all hold almost everywhere. Then consider that for the interval $(0, r-1)$,

$$P1_{[0,r-1]}(x) = \frac{1_{[0,1]}(x)}{r} \sum_{z\in\mathbb{Z}} 1_{[0,r-1]}\left(\frac{x-z}{r}\right).$$

Note that it is only possible that $\frac{x-z}{r} \in [0, r-1]$ and $x \in [0,1]$ when $z = 0$, as $r(r-1) = 1$. Then

$$\begin{aligned} P1_{[0,r-1]}(x) = \frac{1}{r} \cdot 1_{[0,r-1]}\left(\frac{x}{r}\right) &= \frac{1}{r} \cdot 1_{[0,r(r-1)]}(x) \\ &= \frac{1}{r} \cdot 1_{[0,1]}(x) = \frac{1}{r}\left(1_{[0,r-1]}(x) + 1_{[r-1,1]}(x)\right) \end{aligned}$$

almost everywhere. Further,

$$P1_{[r-1,1]}(x) = \frac{1}{r}\left(1_{[r(r-1)-1,r-1]}\right)(x) = \frac{1}{r} \cdot 1_{[0,r-1]}(x).$$

Let $I_0 = (0, r-1)$ and $I_1 = (r-1, 1)$. We define the simple decomposition of I_0, as $d_0 = 1_{I_0}$ and $d_n = 0$ for each $n > 0$. Additionally, $c_{0,0} = c_{0,1} = 1$, and $c_{n,i} = 0$ for $n > 1$. We can also define the simple decomposition of I_1 as follows: let $D_0 = 1_{I_1}$ and $D_n = 0$ for $n > 1$; let $C_{0,0} = 1$ and $C_{n,j} = 0$ for all other pairs $(n, j) \neq (0, 0)$. It is straightforward to verify that these are both well defined. It is of course possible to construct a simple decomposition for each $I \subseteq I_i$, but the argument mainly rests upon the construction of the decompositions of each I_i, and so for now we will act as though this suffices.

For readability, we have used upper and lower case letters to denote the different decompositions; in later sections we will write them as $d_{n,i}$ and $c_{n,i,j}$, where we would have $i, j \in \{0, 1\}$ in this example. As an additional comment on notation, we will write indicator functions such as those above as $1_{[a,b]}$, even when the interval may not be closed. As we are working with equality almost everywhere, this does not affect the statements.

It follows then that

$$P\left(a1_{[0,r-1]} + b1_{[r-1,1]}\right) = \frac{a+b}{r} 1_{[0,r-1]} + \frac{a}{r} 1_{[r-1,1]},$$

and so by induction there are some numbers a_0^m and a_1^m such that

$$P^m(1_{[0,r-1]}) = a_0^m 1_{[0,r-1]} + a_1^m 1_{[r-1,1]},$$

where

$$a_1^{m+1} = \frac{a_0^m}{r}$$

and

$$a_0^{m+1} = \frac{a_0^m + a_1^m}{r}.$$

We may inductively calculate these constants by noting that necessarily $a_0^0 = 1$ and $a_1^0 = 0$. We prove this more generally Lemma 2.7. It is a straightforward exercise to show that a_0^m and a_1^m converge in m. Doing such an argument in general is rather long, and it is much simpler to show that the structure on the a_i^m implies that P is weakly constrictive to the space of functions that can be expressed as

$$\lambda_0 1_{[0,1]} + \lambda_1 1_{[0,r-1]}$$

for $0 \leq \lambda_0 \leq 1$ and $0 \leq \lambda_1 \leq 1/r$. There is some extra work that is necessary in order to show this that we will omit for this section as our goal is only to give an example of constructing decompositions. To complete the argument we would need to construct simple decompositions for each $I \subseteq I_i$ in a similar way, and then use our definition of simple decompositions and the definition of the a_i^m to show that they approach functions of the form $\lambda_0 1_{[0,1]} + \lambda_1 1_{[0,r-1]}$. Once all of this is done, we may apply the Spectral Decomposition Theorem of Lasota and Yorke [9], restated in Theorem 2.5, to obtain asymptotic periodicity.

In Section 2, we will assume that Proposition 1.4 holds, and use it to show that the hypotheses of the Spectral Decomposition Theorem of Lasota and Yorke [9] hold, and thereby we conclude that our main theorem holds. It is also possible to instead improve Proposition 1.4 to give even more structure to the decompositions, and then use the additional structure to directly prove convergence. This was our original approach, but it added another ten to fifteen pages to the proof, and so we have instead adapted the proof to follow a pattern similar to that of [9], though our methods to reach the hypotheses of the Spectral Decomposition Theorem are still quite different. In Section 3, we prove Proposition 1.4 by applying similar ideas as in the above problem, where we demonstrate that $P^n 1_I$ eventually contains I_j in its support; by removing this interval we obtain a c_i and decay in our $\|d_n\|_{\mathrm{L}^1}$.

2 Convergence

Before beginning the main proof, we will quickly apply a reduction to Theorem 1.2.

Theorem 2.1. *If S is a piecewise r-dilation for $r > 2$ such that S' does not change sign on any I_i, then the associated Frobenius-Perron operator is asymptotically periodic.*

Theorem 2.1 implies Theorem 1.2. If S is a piecewise r-dilation, then so too is S^n. Choosing N large enough that $r^N > 2$, we may repartition the I_i so that S^N satisfies the hypotheses of Theorem 2.1. This implies that if P is the Frobenius-Perron operator for S, P^N is asymptotically periodic. Then P must be asymptotically periodic as well. $\square$

Lasota and Mackey's book [11] provides a good overview of their approach to this problem, as well as more details on the definitions that we restate below. We will state them here only for L^1, though they may be generalized to L^p. We call a set $\mathscr{F} \subset L^1$ to be weakly precompact if each sequence $f_n \in \mathscr{F}$ has a weakly convergent subsequence in L^1. The book [11] also provides the logic behind the following lemma.

Lemma 2.2. *For a non-negative function $g \in L^1$, the set of functions $f \in L^1$ such that $|f| \leq g$ is weakly compact.*

To use other theorems, we will quickly define a Markov operator P, following the definition of Lasota and Mackey in [11]. The proof that P is a Markov operator can be found in [11].

Definition 2.3. *An operator $P : L^1 \to L^1$ is a Markov operator if it fulfills the following conditions:*

- *If f is a density function, then so is Pf.*
- *If f is a density function, then $\|Pf\|_{L^1} = \|f\|_{L^1}$.*

Lemma 2.4. *The Frobenius-Perron operator is a Markov operator.*

We call a Markov operator weakly constrictive if there exists a weakly precompact set $\mathscr{F}$ such that for every $f \in L^1$,

$$\lim_{n\to\infty} d(P^n f, \mathscr{F}) = 0\,.$$

We denote the distance between an element $g \in L^1$ and the set $\mathscr{F}$ as $d(g, \mathscr{F})$ and define it as the infimal distance between g and any element of $\mathscr{F}$. We use the result from [8] to simplify the statement of the Spectral Decomposition Theorem so that it applies to weakly constrictive operators. The proof of the Spectral Decomposition Theorem can be found in [9].

Theorem 2.5 (Spectral Decomposition Theorem)**.** *Let P be a weakly constrictive Markov operator. Then there exists an integer b; two sequences of nonnegative functions $\{g_i\}_{i=1}^b \in L^1$ with $\|g_i\|_{L^1} = 1$ and $\{k_i\}_{i=1}^b \in L^\infty$; and an operator $Q : L^1 \to L^1$ such that for all $f \in L^1$, Pf may be written as*

$$Pf(x) = \sum_{i=1}^{b} \left(\int f(x)k_i(x)dx \right) g_i(x) + Q\left(f(x)\right)\,.$$

The functions g_i and operator Q have the following properties:

1. *$g_i(x) \cdot g_j(x) = 0$ for all $i \neq j$, so the functions g_i have disjoint supports.*

2. There exists a permutation $\alpha(i)$ on $\{1, ..., b\}$ such that $Pg_i = g_{\alpha(i)}$.

3. For every $f \in L^1$,

$$\lim_{n\to\infty} \|P^n (Q(f))\|_{L^1} = 0 .$$

This says that Qf acts as a decaying error term. Thus if K is the order of α, then $P^{Kn}f$ converges in L^1.

We have two objectives for this section then. Firstly, we need to construct a weakly precompact set $\mathscr{F}$; this will be given by our simple decompositions. Secondly, we will have to show that $P^n f$ approaches $\mathscr{F}$; this will come from the approximation of measurable functions by step functions.

In this section, we will assume that Proposition 1.4 holds. We fix an $I \subset I_i$ for some i, and let D_m and $C_{m,i}$ be its simple decomposition. We further fix the simple decompositions $d_{n,i}$ and $c_{n,i,j}$, each being a simple decompositions of I_i; while these are not necessarily unique, we will assume that some fixed choice of them is made for the duration of this section.

For clarity, we will use i and j indices to refer to interactions with the I_i given in Theorem 2.1. We use m to refer to the terms that will be present after m applications of P, while n will apply for our decompositions of the I_i.

Definition 2.6. *Let S be an r-dilation for $r > 2$. We say that the constants $\left\{a_{n,i}^m\right\}$ are* ***coefficients of a decomposition of*** $\mathbf{I}$ *if they are defined in the following way. Firstly,*

$$a_{n,i}^0 = 0 \tag{1}$$

for $n \geq 0$ and $i \in \mathcal{I}$. We then induct on m from $m = 0$ to define

$$a_{0,i}^{m+1} = \frac{1}{r}\left(\frac{C_{m,i}}{r^m} + \sum_{j\in\mathcal{I}}\sum_{n=0}^{\infty} c_{n,j,i}a_{n,j}^m\right) \tag{2}$$

for $i \in \mathcal{I}$, and

$$a_{n+1,i}^{m+1} = \frac{a_{n,i}^m}{r} \tag{3}$$

for $n \geq 0$ and $i \in \mathcal{I}$.

Lemma 2.7. *If $a_{n,i}^m$ are the coefficients of a decomposition of I, then for all $m \geq 0$,*

$$P^m 1_I = \frac{D_m}{r^m} + \sum_{i\in\mathcal{I}}\sum_{n=0}^{\infty} a_{n,i}^m \cdot d_{n,i} \,. \tag{4}$$

While Definition 2.6 is strict enough to define a unique object, it is not the only object that could satisfy Lemma 2.7, though there would be little purpose in creating an alternative object, as coefficients of a decomposition have a useful structure for later parts of the proof.

Proof. The case for $m = 0$ is trivial, as

$$P^0 1_I = \frac{D_0}{r^0} + \sum_{i \in \mathcal{I}} \sum_{n=0}^{\infty} 0 \cdot d_{n,i} = D_0 = 1_I$$

by (1). We proceed by induction on m, and aim to show that the lemma holds for $m+1$. Applying the linearity of P and the induction hypothesis, we have that

$$P^{m+1} 1_I = P \left(\frac{D_m}{r^m} + \sum_{i \in \mathcal{I}} \sum_{n=0}^{\infty} a^m_{n,i} \cdot d_{n,i} \right) = P \left(\frac{D_m}{r^m} \right) + \sum_{i \in \mathcal{I}} \sum_{n=0}^{\infty} a^m_{n,i} \cdot P\left(d_{n,i}\right) .$$

By the definition of a simple decomposition, it follows that

$$P^{m+1} 1_I = \frac{D_{m+1} + \sum_{j \in \mathcal{I}} C_{m,j} d_{0,j}}{r^{m+1}} + \sum_{i \in \mathcal{I}} \sum_{n=0}^{\infty} a^m_{n,i} \frac{d_{n+1,i} + \sum_{j \in \mathcal{I}} c_{n,i,j} d_{0,j}}{r} .$$

Recalling our definition of $a^{m+1}_{0,i}$ in (2), we may reorder the sum to be

$$P^{m+1} 1_I = \frac{D_{m+1}}{r^{m+1}} + \sum_{i \in \mathcal{I}} a^{m+1}_{0,i} d_{0,i} + \sum_{i \in \mathcal{I}} \sum_{n=0}^{\infty} \frac{a^m_{n,k}}{r} d_{n+1,k} .$$

The result follows by replacing $a^m_{n,i}/r$ with $a^{m+1}_{n+1,i}$ and then reindexing the last sum for $n' = n+1$. ☐

Lemma 2.8. *If $a^m_{n,i}$ are the coefficients of the decomposition of I, then for all $m, n \geq 0$ and $i \in \mathcal{I}$,*

$$a^m_{n,i} \leq \frac{\|1_I\|_{L^1}}{r^n \|1_{I_i}\|_{L^1}} .$$

Proof. Note that by Lemma 2.4 and Lemma 2.7, we have that

$$\|1_I\|_{\mathrm{L}^1} = \|P^m 1_I\|_{\mathrm{L}^1} = \left\| \frac{D_m}{r^m} + \sum_{i \in \mathcal{I}} \sum_{n=0}^{\infty} a^m_{n,i} \cdot d_{n,i} \right\|_{\mathrm{L}^1} .$$

As all terms are positive, it follows that

$$\|I_I\|_{\mathrm{L}^1} = \frac{\|D_m\|_{\mathrm{L}^1}}{r^m} + \sum_{i \in \mathcal{I}} \sum_{n=0}^{\infty} a^m_{n,i} \cdot \|d_{n,i}\|_{\mathrm{L}^1} \geq a^m_{0,i} \|d_{0,i}\|_{\mathrm{L}^1} = a^m_{0,i} \|1_{I_i}\|_{\mathrm{L}^1} ,$$

and so the lemma holds for $n = 0$. Then if $m > n$, note that by (3),

$$a^m_{n,i} = \frac{a^{m-n}_{0,i}}{r^n} \leq \frac{\|1_I\|_{\mathrm{L}^1}}{r^n \|1_{I_i}\|_{\mathrm{L}^1}} .$$

If $m \leq n$, then notice that by (3) and (1),

$$a_{n,i}^{m} = \frac{a_{n-m,i}^{0}}{r^m} = 0 \leq \frac{\|1_I\|_{\mathrm{L}^1}}{r^n \|1_{I_i}\|_{\mathrm{L}^1}} .$$

Thus for all $n, m \geq 0$, and $i \in \mathcal{I}$, the lemma holds. $\square$

Definition 2.9. *Let $d_{n,i}$ and $c_{n,i,j}$ be fixed simple decompositions of each I_i. We define $g(x)$, our upper bound, as*

$$g(x) = \sum_{i \in \mathcal{I}} \frac{1}{\|1_{I_i}\|_{L^1}} \sum_{n=0}^{\infty} \frac{d_{n,i}(x)}{r^n} .$$

We define $\mathscr{F}$ as the set of all functions $f \in L^1$ with $|f| \leq g$.

The family $\mathscr{F}$ will serve as the pre-compact space to fulfill the hypotheses of Theorem 2.5.

Proof of Theorem 2.1. Let $f \in L^1([0,1])$ be such that $\|f\|_{\mathrm{L}^1} = 1$. Fix a $\delta > 0$; we aim to show that for each sufficiently large m there exist some distribution $f_m \leq g$ such that $\|P^m f - f_m\|_{\mathrm{L}^1} < \delta$. We begin by choosing a step function $\varphi \in L^1([0,1])$ such that $\|f - \varphi\|_{\mathrm{L}^1} < \frac{\delta}{2}$ and $\|\varphi\|_{\mathrm{L}^1} = 1$. Notice that we may write

$$\varphi = \sum_{k=1}^{\ell} \lambda_k 1_{J_k}$$

for some $\lambda_k \in \mathbb{R}^+$, $\ell \in \mathbb{N}$, and $J_k \subset [0,1]$ intervals. We further require that each J_k is contained in some I_i. Notice that

$$\sum_{k=1}^{\ell} \lambda_k \|1_{J_k}\|_{\mathrm{L}^1} = 1 . \tag{5}$$

Let $D_{m,k}$ and $C_{m,k,i}$ form the simple decomposition of each J_k. Further, each has coefficients of decomposition $a_{n,k,i}^{m}$. Choose M large enough that for every $m > M$ and for each k,

$$\frac{\lambda_k \|D_{m,k}\|_{\mathrm{L}^1}}{r^m} < \frac{\delta}{2\ell} . \tag{6}$$

Recall that by Lemma 2.7, for each m and k,

$$\lambda_k \cdot P^m 1_{J_k} = \lambda_k \frac{D_{m,k}}{r^m} + \lambda_k \cdot \sum_{i \in \mathcal{I}} \sum_{n=0}^{\infty} a_{n,k,i}^{m} \cdot d_{n,i} . \tag{7}$$

Then if we define f_m as the second term in (7), so that

$$f_m(x) = \sum_{k=1}^{\ell} \lambda_k \cdot \sum_{i \in \mathcal{I}} \sum_{n=0}^{\infty} a_{n,k,i}^{m} \cdot d_{n,i} ,$$

it follows that

$$P^m\varphi - f_m = \left(\sum_{k=1}^{\ell}\lambda_k P^m 1_{J_j}\right) - f_m = \sum_{k=1}^{\ell}\lambda_k \frac{D_{m,k}}{r^m}\,.$$

Then for $m > M$,

$$\|P^m\varphi - f_m\|_{\mathrm{L}^1} \le \sum_{k=1}^{\ell}\lambda_k \frac{\|D_{m,k}\|_{\mathrm{L}^1}}{r^m} \le \sum_{k=1}^{\ell}\frac{\delta}{2\ell} = \frac{\delta}{2} \tag{8}$$

By Lemma 2.8 and (5), we see that

$$\sum_{k=1}^{\ell}\lambda_k a^m_{n,k,i} \le \frac{1}{r^n\,\|1_{I_i}\|_{\mathrm{L}^1}}\sum_{k=1}^{\ell}\lambda_k\,\|1_{J_k}\|_{\mathrm{L}^1} = \frac{1}{r^n\,\|1_{I_i}\|_{\mathrm{L}^1}}\,.$$

Then it follows that for each $m > M$,

$$f_m(x) = \sum_{i\in\mathcal{I}}\sum_{n=0}^{\infty} d_{n,i}(x)\sum_{k=1}^{\ell}\lambda_k a^m_{n,k,i} \le \sum_{i\in\mathcal{I}}\sum_{n=0}^{\infty}\frac{d_{n,i}(x)}{r^n\,\|I_i\|_{\mathrm{L}^1}} = g(x)\,.$$

We may now examine $\|P^m f - f_m\|_{\mathrm{L}^1}$. Notice that by the triangle inequality,

$$\|P^m f - f_m\|_{\mathrm{L}^1} \le \|P^m f - P^m\varphi\|_{\mathrm{L}^1} + \|P^m\varphi - f_m\|_{\mathrm{L}^1}\ ,$$

and so by the linearity of P and (8), when $m > M$,

$$\|P^m f - f_m\|_{\mathrm{L}^1} \le \|P^m(f-\varphi)\|_{\mathrm{L}^1} + \frac{\delta}{2}\,.$$

By Lemma 2.4, we may reduce the left term to $\|f-\varphi\|_{\mathrm{L}^1}$. It follows from our initial hypothesis on φ that $\|f-\varphi\|_{\mathrm{L}^1} \le \delta/2$ for $m > M$. Thus P is weakly constrictive. Applying the Spectral Decomposition Theorem, it follows that P is asymptotically periodic. □

3 Discretization

Fix a piecewise r-dilation S. We will begin by constructing the Frobenius-Perron operator P.

By Condition 1 of Definition 1.1, the restriction $S_i = S|_{I_i} : I_i \to S(I_i)$ is a bijection. Then we define the almost-inverses as follows.

$$h_i(x) = \begin{cases} (S_i)^{-1}(x) & : x \in S_i(I_i) \\ 0 & : x \notin S_i(I_i) \end{cases} \tag{9}$$

We use them to define the Frobenius-Perron operator associated to S.

$$P : L^1([0,1]) \to L^1([0,1])\,,$$

$$\left(P(d)\right)(x) = \sum_{i\in\mathcal{I}} |h_i'(x)| \cdot d\left(h_i(x)\right) \,.$$

By the third condition of Definition 1.1, this is well-defined almost everywhere.

In this construction, P can move inside of integrals according to the following lemma.

Lemma 3.1. *If $d \in L^1(S)$ is non-negative, and*

$$d(x) = \int f_t(x)dt$$

for a family of distributions f_t, then

$$P(d)(x) = \int P(f_t)(x)dt\,.$$

Proof. Notice that

$$Pd(x) = \sum_{i=0}^{\infty} |h_i'(x)| \cdot d\left(h_i(x)\right) = \sum_{i=0}^{\infty} |h_i'(x)| \int f_t\left(h_i(x)\right)dt\,.$$

We now exchange the order of the sum and integral and then apply the definition of P to see that

$$Pd(x) = \int \left(\sum_{i\in\mathcal{I}} |h_i'(x)| \cdot f_t(h_i(x)) \right) dt = \int P f_t(x)dt\,.$$

□

We restate a remark from the introduction here.

Remark. *If $I \subseteq I_i$ is an interval, then so is $S(I)$. Further, because h_i is monotonic on I_i and $h_j = 0$ on I_i if $j \neq i$, it follows that the level set of $P(1_I)(x)$,*

$$J_t = \{x : P(1_I)(x) > t\}\ ,$$

is an interval as well.

As $P(1_I)(x) \leq \frac{1}{r}$,

$$P(1_I)(x) = \int_0^{1/r} 1_{J_t}(x)dx = \frac{1}{r}\int_0^1 1_{J_s}(x)ds\,.$$

We will often use a substitution for $t' = t/r$ so that our integral runs from 0 to 1, and we extract a $1/r$ factor from the integral.

Combining Lemma 3.1 and the above remark, if I is an interval such that for every $m \leq n$, $S^m(I) \subset I_{i_m}$ for some i_m, then there exists a collection of intervals I_v, $v \in [0,1]^n$, such that

$$P^n d(x) = \frac{1}{r^n}\int_{v\in[0,1]^n} 1_{I_v}(x)dv\,.$$

To construct this more exactly, it is useful to introduce definitions that let us work inductively on $[0,1]^n$ more easily. We begin with a definition for a countable tree, which we then expand to a "continuous" tree. We will assume that $\mathcal{I} = \{0, 1, ..., n\}$ for some $n \in \mathbb{N}$.

Definition 3.2. *We call $T_{\mathcal{I}}$ to be the tree over $\mathcal{I}$ and define it as the collection of all $(n+1)$-tuples of non-negative integers $(i_k)_{k=0}^n$ that have $i_0 = 0$ and $i_k \in \mathcal{I}$.*

It is often easiest to write an element $e \in T_{\mathcal{I}}$ explicitly as $e = (0, i_1, i_2, ...i_n)$. We denote the root element of the tree, the unique element of size 0, as (0). We recommend thinking of i_0 as representing beginning at the root of the tree and each i_k as the choice of children that leads to the current position. Below, the reader can see an example of a tree over $\{0, 1\}$ written in our notation.

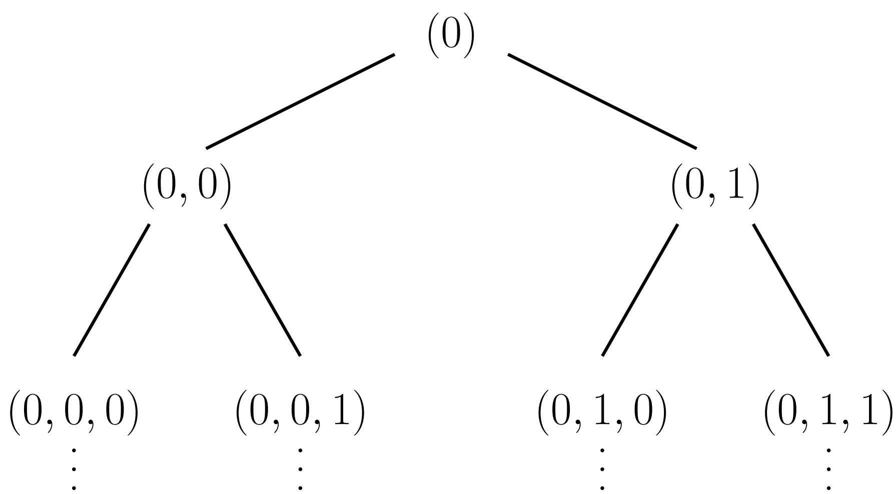

Figure 1: tree notation

Definition 3.3. *If $e \in T_{\mathcal{I}}$, we call $|e|$ the size of e, where if $e = (i_k)_{k=0}^n$, then*

$$|e| = n$$

The choice to begin counting size at 0 instead of 1 will allow us to have powers $r^{|e|}$ in a natural way.

Definition 3.4. *If $e, f \in T_{\mathcal{I}}$, $e = (i_k)_{k=0}^n$, and $f = (j_k)_{k=0}^m$, then we say that $e \subseteq f$ if $n \leq m$ and $i_k = j_k$ for each $0 \leq k \leq n$*

Definition 3.5. *If $e \in T_{\mathcal{I}}$ and $|e| > 0$, we define $\overline{e}$ to be the unique element so that $\overline{e} \subset e$ and $|\overline{e}| + 1 = |e|$.*

Definition 3.6. *If $e, f \in T_{\mathcal{I}}$, $e = (i_k)_{k=0}^n$, and $f = (j_k)_{k=0}^m$ then we say that their conjunction $ef \in T_{\mathcal{I}}$ is $(\ell_k)_{k=0}^{n+m}$, where if $k \leq n$, $\ell_k = i_k$, and for $k > n$, $\ell_k = j_{k-n}$. It is helpful to note that written out,*

$$ef = (0, i_1, \ldots, i_n, j_1, \ldots, j_m).$$

Effectively, a conjunction treats the node e as the root of a sub-tree and then chooses f in that sub-tree. This is why we have to remove the first 0 element from f.

Definition 3.7. *If $e \in T_{\mathcal{I}}$, let $D^h(e) = \{f \in T_{\mathcal{I}} : e \subseteq f, |f| = |e| + h\}$*

We will often write $D^1(e)$ simply as $D(e)$ and call them the descendants of e. Note that for all $f \in D(e)$, $\overline{f} = e$.

We use similar notation for the continuous tree over $[0,1]$, which we denote $T_{[0,1]}$, the collection of all $n+1$ tuples of real numbers $(t_0, t_1, ..., t_n)$ that fulfill

- $t_0 = 0$
- $t_k \in [0,1]$ for all $0 \leq k \leq n$.

For the sake of consistency, we will denote the elements of $T_{\mathcal{I}}$ with lower-case letters starting from e and the elements of $T_{[0,1]}$ with upper-case letters starting from E.

Definition 3.8. *We say that a non-trivial interval I, which is contained in some I_j, has a* ***complicated decomposition***

$$(T_{\mathcal{I}}, T_{[0,1]}, I_{e,E}, c_{e,E,i})$$

if $c_{e,E,i} \geq 0$, the $I_{e,E}$ are intervals whenever $|e| = |E|$, and $I_{(0),(0)} = I$. We additionally require the following two conditions.

1.

$$P(1_{I_{e,E}})(x) = \frac{1}{r}\left(\int_0^1 \sum_{i\in\mathcal{I}} 1_{I_{e(0,i),f(0,t)}}(x)dt\right) + \frac{1}{r}\sum_{i\in\mathcal{I}} c_{e,E,i} \cdot 1_{I_i}$$

2.

$$\sum_{e\in T_k} \frac{1}{r^{|e|}}\left(\int_{|E|=|e|} \left\|1_{I_{e,E}}\right\|_{L^1} dE\right) < \infty$$

Above we use $\int_{|E|=n} \|1_{e,E}\|_{L^1} dE$ to be the integral over $[0,1]^n$, where for a vector $v = (t_1, ..., t_n) \in [0,1]^n$, $E = (0, t_1, ...t_n)$.

Complicated decompositions allow us to make use of Lemma 3.1, and thus are easier to initially construct. However, due to the parallels between Conditions 1 and 2 for this definition and Definition 1.3, we are able to obtain the following Lemma.

Lemma 3.9. *If I has a complicated decomposition, then it also has a simple decomposition.*

Remark. *Both complicated and simple decompositions are not unique. In fact, one can obtain even stronger conditions on them by refining the construction. For example, it can be proved that one can construct a set $G \subset \mathcal{I}$ so that for each $i \in G$ a simple decomposition of I_i can be made so that $c_{i,n,j} = 0$ if $j \neq i$, and for each $i \notin G$, $c_{i,n,j} = 0$ if $j \notin G$. As this is not necessary to prove our main result, we do not delve further into this fact.*

Proof. Let $(T_{\mathcal{I}}, T_{[0,1]}, I_{e,E}, c_{e,E,i})$ be a complicated decomposition of I. Note that $I_{(0),(0)} = I$. Then we define

$$d_n(x) = \sum_{|e|=n} \int_{|E|=n} 1_{I_{e,E}}(x) dE$$

and

$$c_{n,i} = \sum_{|e|=n} \int_{|E|=n} c_{e,E,i} dE \, .$$

Then notice that by Definition 3.8 and Lemma 3.1,

$$Pd_n(x) = \sum_{|e|=n} \int_{|E|=n} \left(\frac{1}{r} \left(\int_0^1 \sum_{i \in \mathcal{I}} 1_{I_{e(0,i),E(0,t)}}(x) dt \right) + \frac{1}{r} \sum_{i \in \mathcal{I}} c_{e,E,i} \cdot 1_{I_i}(x) \right) dE \, .$$

It is clear that the last terms, once separated, represent the $c_{n,i}$.

We can also note that summing over $|e| = n$, and then $e(0,i)$ is the same as summing over all f such that $|f| = n+1$. Similarly we reduce the integrals to an integral over $|F| = n+1$ and conclude that

$$Pd_n = \frac{1}{r} \left(\sum_{|f|=n+1} \int_{|F|=n+1} 1_{I_{f,F}}(x) dF \right) + \frac{1}{r} \left(\sum_{i \in \mathcal{I}} 1_{I_i} \sum_{|e|=n} \int_{|E|=n} c_{e,E,i} dE \right) \, .$$

Then directly applying our definitions of d_n and $c_{n,i}$, it is clear that

$$Pd_n = \frac{1}{r} \left(d_{n+1} + \sum_{i \in \mathcal{I}} c_{n,i} \cdot 1_{I_i} \right) \, .$$

This shows that condition 1 of Definition 1.3 is fulfilled. Condition 2 is immediate from the second condition of Definition 3.8. □

Because of Lemma 3.9 in order to prove Proposition 1.4, it suffices to prove the following.

Proposition 3.10. *Let I be a non-trivial interval contained in some I_i. Then I has a complicated decomposition $(T_{\mathcal{I}}, T_{[0,1]}, I_{e,E}, c_{e,E,i})$.*

Proof. Define $I_{(0),(0)} = I$, and proceed with an inductive contstruction, assuming that $I_{e,E}$ is defined for some $|e| = |E|$. We define

$$J_{e,E,t} = \left\{ x : P(1_{I_{e,E}})(x) \geq \frac{t}{r} \right\} \, .$$

Note that if $I_{e,E}$ is an interval entirely contained in some I_j, then $J_{e,E,t}$ is an interval as well. We now define $I_{e(0,i),E(0,t)}$ as well as constants $C_{e,E,i,t}$ as follows.

If $I_i \subseteq J_{e,E,t}$, then

$$I_{e(0,i),E(0,t)} = \emptyset$$

and

$$C_{e,E,i,t} = 1\,.$$

Then if I_i is not entirely contained in $J_{e,E,t}$, we define

$$I_{e(0,i),E(0,t)} = J_{e,E,t} \bigcap I_i$$

and

$$C_{e,E,i,t} = 0\,.$$

In this construction, whenever we can "remove" a copy of I_i from $J_{e,E,t}$. When we do, we create the decay to fulfill Condition 2 of Definition 3.8 in exchange for obtaining a term 1_{I_i}.

Finally, we define

$$c_{e,E,i} = \int_0^1 C_{e,E,i,t} dt\,.$$

It is clear from the above definitions that each $I_{e(0,i),E(0,t)}$ is an interval contained inside I_i. Further, since $|S'| \geq r$, it follows that $\left|P1_{I_{e,E}}\right| \leq 1/r$, and so we need only consider the $J_{e,E,t}$ for $0 \leq t \leq 1$. Then it follows that Condition 1 of Definition 3.8 holds, as

$$\frac{1}{r}\left(\int_0^1 \sum_{i \in \mathcal{I}} 1_{I_{e(0,i),E(0,t)}}(x) dt\right) + \frac{1}{r}\sum_{i \in \mathcal{I}} c_{e,E,i} \cdot 1_{I_i}(x)$$

is equal to

$$\frac{1}{r}\int_0^1 \left(\sum_{i \in \mathcal{I}} 1_{I_{e(0,i),E(0,t)}}(x) + C_{e,E,t} \cdot 1_{I_i}(x)\right) dt\,,$$

which in turn we may write as

$$\frac{1}{r}\int_0^1 \sum_{i \in \mathcal{I}} 1_{J_{e,E,t}}(x) \cdot 1_{I_i}(x) dt = \frac{1}{r}\sum_{i \in \mathcal{I}} \int_{I_i} 1_{J_{e,E,t}}(x) dt = \frac{1}{r}\int_0^1 1_{J_{e,E,t}}(x) dt\,.$$

By definition of the level sets, this is precisely $P1_{e,E}(x)$.

In order to examine Condition 2 of Definition 3.8, we must first notice that $J_{e,E,t}$ is an interval. Since the union of the closures of the I_i cover $[0,1]$, there may only be two I_i such that $I_i \cap J_{e,E,t} \neq \emptyset, I_i$. If we fix t, we may call these two intervals I_{i_1} and I_{i_2}. Then for all other $i \neq i_1, i_2$, it follows that $I_{e(0,i),E(0,t)} = \emptyset$. Notice that if $I_{e,E} = \emptyset$ and $e \subset f$, $E \subset F$, then $I_{f,F} = \emptyset$ as well.

Then for a fixed E with $|E| = n$, it follows that out of all choices of $e \in T_\infty$, at most 2^n of them have that $I_{e,E} \neq \emptyset$. We define $1_{I_{e,E} \neq \emptyset}$ to be 1 if $I_{e,E}$ is nonempty and 0 otherwise. From here we notice that for each $n \geq 0$,

$$\sum_{|e|=n}\left(\int_{|E|=n} \left\|1_{I_{e,E}}\right\|_{\mathrm{L}^1} dE\right) \leq \int_{|E|=n} \sum_{|e|=n} 1_{I_{e,E} \neq \emptyset} dE\,. \tag{10}$$

By our remark above, we may bound (10) above by

$$\int_{|E|=n} 2^n dE\,.$$

Since each element E with $|E| = n$ is a vector in $[0,1]^n$, the measure of the space is 1, and so (10) is bounded by 2^n. Then

$$\sum_{e\in T_k} \frac{1}{r^{|e|}} \left(\int_{|E|=|e|} \left\|1_{I_{e,E}}\right\|_{\mathrm{L}^1} dE \right) \le \sum_{n=0}^{\infty} \frac{2^n}{r^n} < \infty$$

since $r > 2$.

$\square$

Combining Proposition 3.10 with Lemma 3.9 show that each interval fulfilling the conditions of Proposition 1.4 has a simple decomposition; the bound on $\|d_n\|_{\mathrm{L}^1}$ follows by bounds obtained near the end of the proof of Proposition 3.10.

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