1785

Sponsored by: UGA Math Department

WRITTEN TEST, 25 PROBLEMS / 90 MINUTES / 250 POINTS

October 21, 2023

WITH SOLUTIONS

Problem 1. What is the sum of the squares of the digits of the square of the sum of the digits of 2023?

(A) 10 (B) 31 (C) 61 (D) 64 (E)$^\heartsuit$ 97

Answer. 97

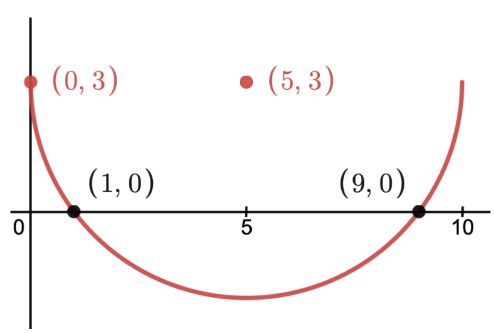
Solution. Following the instructions, we get successively $2+0+2+3=7$, $7^2=49$, $4^2+9^2=97$.

Problem 2. Let C be a circle of radius 5, centered at $(5,3)$. The parabola of equation $y=f(x)$ shares both its x and y intercepts with C. What is $f(2)$?

(A)$^\heartsuit$ $-\frac{7}{3}$ (B) $-\frac{5}{3}$ (C) -2 (D) $-\frac{2}{3}$ (E) $-\frac{2}{3}$

Answer. $-\frac{7}{3}$

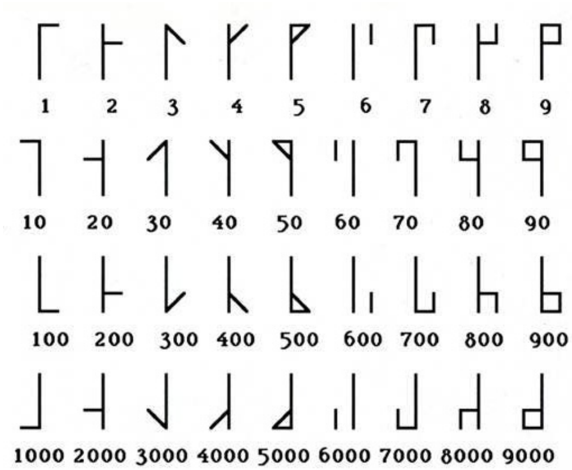
Solution.



Since the radius is 5 and the center's abscissa is 5 as well, the y-axis axis is tangent to the circle at $(0,3)$; this is our y-intercept. As the center lies 3 units above the x axis and since the radius is 5, by Pythagoras' theorem we conclude that the circle

intersects that axis at $x = 5 \pm 4$, i.e. at the points $(1, 0)$ and $(9, 0)$.
Now that we know the roots of the parabola, we can deduce that it is of the form $y(x) = k(x-1)(x-9)$. The y-intercept, 3, then settles the value of k so that $y(x) = \frac{1}{3}(x-1)(x-9)$ whence $y(2) = -\frac{7}{3}$

Problem 3. During the Middle Ages, Cistercian monks developed an interesting additive numeration system where each number from 1 to 9999 could be expressed as a single symbol. Their convention is illustrated in the table below:



For example, 2023 would be represented by , 1453 by  and 732 by . If X is the largest multiple of 4 whose Cistercian notation is invariant under a 180^o rotation, what is X?

(A) 6996 (B) 8080 (C) 8008 (D)$^\heartsuit$ 8888 (E) 9696

Answer. 8888

Solution. Rotating a cistercian symbol by 180^o permutes the units with the thousands and the tens with the hundreds. Such a number is thus invariant under rotation if and only if it is of the form *abba* where a and b represent digits. A number is divisible by 4 if and only if its last two digits are divisible by 4. Given the palindromic nature of the number we are looking for, we seek for the multiple of four with the largest possible unit digit and then with the largest possible ten digit. This number is clearly 88 as multiples of 4 are even, hence $X = 8888$.

Problem 4. Amongst four friends, Alice, Bob, Charly and Donna, each person either always lies or always tell the truth. One evening, they make the following statements:

Alice - Bob is a liar!
Charly - Alice is a liar.

Donna - Alice and Charly are both liars. Bob is a liar!

Who are the liars?

(A) Alice & Bob (B) Alice & Charlie (C)$^\heartsuit$ Alice & Donna
(D) Bob & Donna (E) Charlie & Donna

Answer. Alice and Donna

Solution. Assume Donna says the truth. Then, Alice is a liar (according to Donna's statement) so Bob must tell the truth (since Alice is liar). However, this contradicts Donna's statement. Therefore, our assumption is wrong and Donna is a liar. Hencefort, Bob tells the truth (since Donna is liar) and so Alice is also liar. Lastly, Charlie tells the truth as Alice is liar. As a consequence, Alice and Donna are the two liars.

Problem 5. Alice and Bob are bored and decide to play a game. The players alternate taking turns and add 1 or 2 (to their liking) to the number that the previous player has given. The first player who says the number "n" wins. If both players play with perfect strategy, which of the following n ensures that "Alice wins the game?

(A) 26 (B) 30 (C)$^\heartsuit$ 34 (D) 38 (E) 42

Answer. 34

Solution. If Alice starts with the number "1", it is guaranteed that Alice can say "4" in her second turn (notice that it does not matter whether Bob says "2" or "3"). If Alice says "4" in the second turn, it is guaranteed that she can say "7" in her third turn. Therefore, Alice can go with the following numbers "$1, 4, 7, 10, 13, \ldots$" all have remainder 1 on dividing by 3. Hence, if n is 34, Alice is guaranteed to win the game.

Problem 6. An ant starts on one corner of a cube, and randomly chooses an edge of the cube to walk across. After reaching the next corner, the ant once again chooses one of the three available edges to walk across. If the ant continues in this fashion, after walking across edges of the cube 6 times, what is the probability of the ant ending on the vertex where it started?

(A)$^\heartsuit$ $\frac{61}{243}$ (B) $\frac{17}{81}$ (C) $\frac{1}{8}$ (D) $\frac{17}{64}$ (E) $\frac{41}{162}$

Answer. $\frac{61}{243}$

Solution. For convenience, we will consider the vertices of our cube to be the points of the form (x, y, z) where $x, y, z \in \{0, 1\}$. We also assume that the ant starts at the origin. Let $f(x, y, z) = x + y + z$, and note that this is the length of the shortest path from $(0, 0, 0)$ to (x, y, z) along edges of the cube. Instead of keeping track of exactly where the ant goes on the cube, we will keep track of the value of $f(x, y, z)$. Let A_n be the event that after n steps, the ant lands at a point where $f(x, y, z) = 0$, i.e. the ant lands on the origin. Similarly, let B_n, C_n, D_n be the events that the ant lands on a point where $f(x, y, z)$ is equal to $1, 2, 3$ respectively.

When the ant is at the origin, it is guaranteed to travel to a point where $f(x, y, z) = 1$. When it is at a point where $f(x, y, z) = 1$, it has a $\frac{1}{3}$ chance of traveling back to the origin, and a $\frac{2}{3}$ chance of going to a point where $f(x, y, z) = 2$. When it is at a point where $f(x, y, z) = 2$, it has a $\frac{2}{3}$ chance of going to a point where $f(x, y, z) = 1$, and a $\frac{1}{3}$ chance of going to a point where $f(x, y, z) = 3$, i.e. the point $(1, 1, 1)$. When the ant is at the point $(1, 1, 1)$, it is guaranteed to go to a point where $f(x, y, z) = 2$.

Thus, we have the recursion $A_{n+1} = \frac{1}{3}B_n$, $B_{n+1} = A_n + \frac{2}{3}C_n$, $C_{n+1} = \frac{2}{3}B_n + D_n$, and $D_{n+1} = \frac{1}{3}C_n$. Moreover, we have the initial state $(A_0, B_0, C_0, D_0) = (1, 0, 0, 0)$. Therefore, we can find the sequence (A_n, B_n, C_n, D_n) for $n = 0, ..., 5$:

$$(1,0,0,0),(0,1,0,0),(\frac{1}{3},0,\frac{2}{3},0),(0,\frac{7}{9},0,\frac{2}{9}),(\frac{7}{27},0,\frac{20}{27},0),(0,\frac{61}{81},0,\frac{20}{81}).$$

Since $B_5 = \frac{61}{81}$, we conclude that $A_6 = \frac{61}{243}$.

Problem 7. Starting with a regular 2023-gon, suppose that you choose some number of pairs of vertices and draw the diagonals between them, ensuring that the diagonals do not intersect. What is the least number of diagonals you can draw so that it is not possible to include any additional non-intersecting diagonals?

(A) 1612 (B) 1774 (C) 1922 (D) 1993 (E)$^\heartsuit$ 2020

Answer. 2020

Solution. It is clear that one can draw 2020 diagonals all starting from the same vertex, and after that it is impossible to add any more non-intersecting diagonals. The question is, can we achieve this with less than 2020 diagonals?
We can see that the answer is no by induction. We claim that in fact no matter how one chooses to draw the diagonals in an n-gon, the process will terminate after drawing $n-3$ diagonals. This is clearly true for $n = 3, 4$. Suppose that for all k-gons, $k < n$, the process of drawing non-intersecting diagonals always terminates after $k-3$ diagonals have been drawn. Now suppose that we draw one diagonal in an n gon, dividing it into a k-gon and an $(n-k+2)$-gon. These sub-polygons can accept $k-3$ diagonals and $n-k-1$ diagonals respectively, for a total of $n-4$ diagonals. Counting the one diagonal we drew to start, we see the process always terminates after drawing $n-3$ diagonals, and the claim is proved by induction.

Problem 8. Let x,y and z be real numbers and assume that $\frac{xyz}{y+z} = -1$, $\frac{xyz}{x+z} = 1$ and $\frac{xyz}{x+y} = 2$; which of the following could be the value of xyz?

(A)$^\heartsuit$ $-\frac{8}{\sqrt{5}}$ (B) $-\frac{4}{5\sqrt{3}}$ (C) $-\frac{1}{\sqrt{2}}$ (D) $\frac{5}{2\sqrt{7}}$ (E) $\frac{3}{\sqrt{2}}$

Answer. $-\dfrac{8}{\sqrt{5}}$[1]

Solution. Taking the reciprocals of the above we can write $\frac{x+y}{xyz} = \frac{1}{2}$, $\frac{y+z}{xyz} = -1$ and $\frac{x+z}{xyz} = 1$.
Then, $\frac{x+y}{xyz} + \frac{y+z}{xyz} + \frac{x+z}{xyz} = \frac{2(x+y+z)}{xyz} = \frac{1}{2}$ which implies that $\frac{x+y+z}{xyz} = \frac{1}{4}$. Therefore $\frac{x}{xyz} = \frac{1}{4} + 1 = \frac{5}{4}$, $\frac{y}{xyz} = \frac{1}{4} - 1 = -\frac{3}{4}$, $\frac{z}{xyz} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$ and thus $\frac{xyz}{(xyz)^3} = \frac{15}{64} \implies \frac{1}{(xyz)^2} = \frac{15}{64}$
We can conclude that $xyz = \pm\frac{8}{\sqrt{5}}$.

Problem 9. On planet Zglub, everything is made out of 4 fundamental particles: the archon $\bigcap$, the dawgon Δ, the bullon $\bigotimes$ and the touchdon $\bigsqcup$. These particles are very unstable and after each collision the total number of particles decreases by 1 according to the following rules :

1. An archon always gets absorbed : $\bigcap + X \rightarrow X$, where X is any of the four particles.
2. Two identical particles transform into an archon : $X + X \rightarrow \bigcap$.
3. A touchdon transforms a dawgon into a bullon, and transforms a bullon into a dawgon : $\Delta + \bigsqcup \rightarrow \bigotimes$ and $\bigotimes + \bigsqcup \rightarrow \Delta$.
4. A dawgon and a bullon give a touchdon $\Delta + \bigotimes \rightarrow \bigsqcup$.

Inside his lab, a scientist has created a small scale model of the planet by putting together 441 archons ($\bigcap$), 673 dawgons (Δ), 431 bullons ($\bigotimes$) and 478 touchdons ($\bigsqcup$). After 2022 collisions, only one particle survives the experiment. What is that terminal particle?

(A) $\bigotimes$ (B) Δ (C) $\bigcap$ (D)$^{\heartsuit}$ $\bigsqcup$

Answer. $\bigsqcup$

Solution. We can think of particle collision as a binary operation on the set of particles, $\{\bigcap, \Delta, \bigotimes, \bigsqcup\}$. The description of the problem suggests that this operation is commutative and associative. This can be easily checked by writing the Cayley table of the (*) operation representing particle collision:

$+$	$\bigcap$	Δ	$\bigotimes$	$\bigsqcup$
$\bigcap$	$\bigcap$	Δ	$\bigotimes$	$\bigsqcup$
Δ	Δ	$\bigcap$	$\bigsqcup$	$\bigotimes$
$\bigotimes$	$\bigotimes$	$\bigsqcup$	$\bigcap$	Δ
$\bigsqcup$	$\bigsqcup$	$\bigotimes$	Δ	$\bigcap$

[1] The problem was removed as the initial answer was erroneously recorded as $-\frac{8}{3\sqrt{5}}$.

These properties mean that there many ways to compute the final particle.

This being said, the simplest is to realize that rule (2) reduces any initial number of a given particle, $X + \cdots + X$, to X if they are in odd number and to $\bigcap$ if they are in even number.

by putting together 441 archons ($\bigcap$), 673 dawgons (Δ), 431 bullons ($\bigotimes$) and 478 touchdons ($\bigsqcup$).

Therefore, the initial configuration of 441$\bigcap$, 673Δ, 431$\bigotimes$ and 478$\bigsqcup$ reduces to a single particle of each type. Now $\bigcap + \Delta \to \bigotimes$ and $\bigotimes + \bigotimes \to \bigsqcup$ and, finally, $\bigsqcup + \bigsqcup \to \bigsqcup$. In other words, the last two particles eventually collide to form a touchdon ! $\bigsqcup$!

Remark: the particles behave like $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ after performing the following identification: $\bigcap = (0,0)$, and any bijection between the remaining three particles and $(0,1)$, $(1,1)$ and $(1,0)$.

Problem 10. Moody the donkey is very stubborn. He only moves eastwards (E) or northwards (N). Every morning, he leaves from his stable and moves 1/2 mile either (E) or (N), then he reassesses the situation and walks 1/4 mile either (E) or (N), at that point, he decides to move 1/8th of mile either (E) or (N), etc. That day, Moody decides to alternate, he goes first (E), then (N), then (E), etc. How far from home will he end up?

(A) $\frac{\sqrt{2}}{3}$ (B) $\frac{\sqrt{3}}{2}$ (C)$^\heartsuit$ $\frac{\sqrt{5}}{3}$ (D) 1 (E) 2

Answer. $\frac{\sqrt{5}}{3}$

Solution. Moody's overall horizontal displacement is $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$. His total vertical displacement is half of that, i.e. $\frac{1}{4} + \frac{1}{16} + \cdots = \frac{1}{3}$. Using Pythagoras, we get a total displacement of $\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \dfrac{\sqrt{5}}{3}$.

Problem 11. Today, Moody the donkey (see Problem 10) does not feel adventurous. He picks a path so as to stay as close as possible to home. How far from home (in miles) will he land?

(A)$^\heartsuit$ $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{3}$ (C) $\frac{\sqrt{2}}{3}$ (D) $\frac{\sqrt{3}}{2}$ (E) $\frac{\sqrt{5}}{3}$

Answer. $\frac{\sqrt{2}}{2}$

Solution. Arguing as above, the total horizontal displacement h and the total vertical displacement v satisfy $v + h = 1 = \frac{1}{2} + \frac{1}{4} + \cdots$. The distance from home is thus $\sqrt{v^2 + (1-v)^2} = \sqrt{2v^2 - 2v + 1}$ which is minimal at the vertex of the radicand, i.e. when $v = -\frac{-2}{4} = \frac{1}{2}$. In that case, we get a distance of $\frac{\sqrt{2}}{2}$.

Problem 12. . Arrange the integers in a pyramidal form as follows:

$$\begin{array}{c} 1 \\ 2\ 3 \\ 4\ 5\ 6 \\ 7\ 8\ 9\ 10 \\ 11\ 12\ 13\ 14\ 15 \end{array}$$

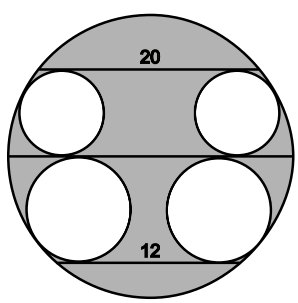
The list goes on with every time n entries on row n. What is the sum of the entries on the 10th row?

(A) 300 (B) 369 (C)$^\heartsuit$ 505 (D) 561 (E) 565

Answer. 505

Solution. There are n entries in row n, hence before row n, there are $1+\ldots+(n-1) = \frac{n(n-1)}{2}$ entries. This means that row 10 begins with $\frac{10\cdot 9}{2} + 1 = 46$. The last entry of that row is 55; we have a total of $\frac{10}{2} \times (46 + 55) = 505$.

Problem 13. Consider the figure below representing a configuration of two chords parallel to a same diameter and four circles, pairwise tangent to the diagonal and one of the two chords. If the length of the chords are as above, what is the area of the shaded region?

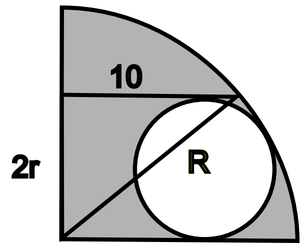


(A) 34π (B) 36π (C) 64π (D)$^\heartsuit$ 68π (E) 100π

Answer. 68π

Solution.

- Let us focus on the top right quadrant of the circle.



From Pythagoras' theorem follows that $R^2 = 100 + 4r^2$ where R denotes the radius of the outside circle and r the radius of the inner one. The shaded area in that quadrant is thus $\frac{\pi}{4}(100 + 4r^2) - \pi r^2 = 25\pi$. The area is the top left quadrant is of course identical and, *mutatis mutandis*, we get an area of 9π for each of the bottom two quadrants. All in all, the shaded area is thus 68π.

- An alternative solution is to first focus on the top half of the circle. From the question, the answer seems to be independent of the outer radius and inner radius (as those could vary) and only depend on the chord length. One could thus imagine a degenerate (extreme) case where the inner circles shrink to a point and the chord becomes a diameter. We then are reduced to compute the half area of a disk of diameter 20, i.e. 50π. Similary, the lower half of the picture yields an area of 18π.

Problem 14. Evaluate the sum $\sum_{n=-3}^{4} \log(n + \sqrt{n^2 + 9})$, i.e.

$$\log(-3 + \sqrt{3^2 + 9}) + \log(-2 + \sqrt{2^2 + 9}) + \cdots + \log(4 + \sqrt{4^2 + 9}).$$

(A) 0 (B) $2\log(3)$ (C) $3\log(3)$ (D) $8\log(3)$ (E)$^\heartsuit$ $9\log(3)$

Answer. $9\log(3)$

Solution. It useful to pair elements in the sum with opposite indices as

$$\log(-n + \sqrt{(-n)^2 + 9}) + \log(n + \sqrt{n^2 + 9}) = \log(-n^2 + (\sqrt{n^2 + 9})^2) = \log(9).$$

Beware that the $n = 0$ term is simply $\log(3)$. Finally, the last term, $n = 4$ simplifies to $\log(4+\sqrt{4^2 + 9}) = \log(9)$. All in all, the sum adds up to $3\log(9)+\log(9)+\log(3) = 9\log(3)$.

Problem 15. The distinct real numbers a, b and c all satisfy the same equality :

$$a = \sqrt[3]{37a + 84}, \quad b = \sqrt[3]{37b + 84}, \quad c = \sqrt[3]{37c + 84}.$$

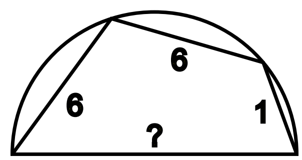
What is the sum of the digits of their product, abc?

(A) 3 (B)$^\heartsuit$ 12 (C) 5 (D) 6 (E) 7

Answer. 12

Solution. The equations above simply mean that a, b and c are the roots of the polynomial $x^3 - 37x - 84$. The constant coefficient in a polynomial is the product of the negative of its roots; this can be seen by expanding $(x - a_1) \cdots (x - a_n)$. We can thus conclude conclude that $abc = 84$ and the sum of the digits is thus 12.

Problem 16. What is the diameter of the semicircle below given that its endpoints can be joined by three connected chords whose lengths are 6, 6 and 1, as shown?



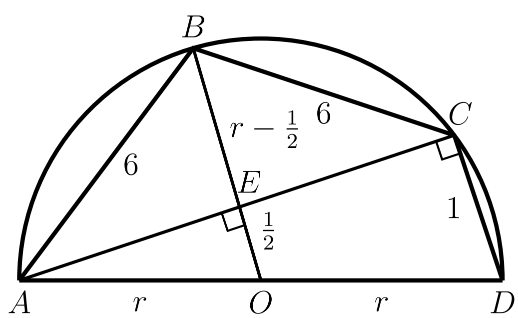
(A) 6 (B) 7 (C) 8 (D)$^\heartsuit$ 9 (E) 10

Answer. 9

Solution.

Option 1 (Similar triangles)

For what follows below, it is useful to label some points, so that we can name some of the relevant segments, angles, and polygons. Let O be the center of the semicircle, let A, B, C, D be the points on the circumference, starting from the left going in clockwise order. Let E be the intersection of BO and AC.



Now let r be the radius of the semicircle.

First, observe that $OABC$ is a kite with $AB = BC = 6$, $OA = AC = r$. This implies that the diagonals are perpendicular, i.e., $AC \perp BO$, and so $\angle AEO = \angle AEB = 90°$. Also, $\triangle ACD$ is a right triangle, as it is inscribed in a semicircle whose diameter is its hypotenuse. We then have that $\triangle AEO \sim \triangle ACD$. Since $AO = r$ and $AD = 2r$, the ratio of similarity is 2; we then must have that $EO = 1/2$. In turn, $BE = r - 1/2$.

Observe also that both $\triangle AEO$ and $\triangle AEB$ are right triangles. Using the Pythagorean theorem, we can now compute AE in two different ways: we have that

$$AE^2 = r^2 - \left(\frac{1}{2}\right)^2 = 6^2 - \left(r - \frac{1}{2}\right)^2.$$

After simplifying and expanding, we have

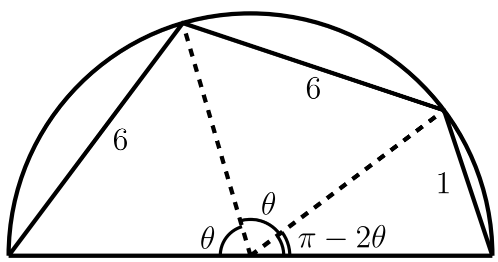
$$\begin{aligned} r^2 - \frac{1}{4} &= 36 - \left(r^2 - r + \frac{1}{4}\right) \\ 2r^2 - r - 36 &= 0 \\ (2r-9)(r+4) &= 0. \end{aligned}$$

Since $r > 0$, we must have $r = 9/2$, and so the diameter is $2r = 9$.

Option 2 (Some trigonometry)

As in Solution 1, denote by r the radius of the semicircle; the quantity we are looking for is then $2r$.

Let θ be the central angle of each arc subtended by a chord of length 6, so that the angle of the arc subtended by the chord of length 1 is $\pi - 2\theta$.



By the cosine law, we have:

$$\begin{aligned} 6^2 &= 2r^2 - 2r^2\cos(\theta) = 2r^2(1-\cos(\theta)) \\ 1^2 &= 2r^2 - 2r^2\cos(\pi - 2\theta) = 2r^2(1+\cos(2\theta)) \end{aligned}$$

Divide the second equation by the first to get:

$$\begin{aligned} \frac{1}{36} &= \frac{1+\cos(2\theta)}{1-\cos\theta} \\ \frac{1}{36} &= \frac{2\cos^2(\theta)}{1-\cos(\theta)} \\ 72\cos^2\theta + \cos\theta - 1 &= 0 \\ (9\cos\theta - 1)(8\cos\theta + 1) &= 0 \end{aligned}$$

Since $0 < \theta < \pi/2$, we must have $\cos\theta = 1/9$. Substituting this back into the first equation we have $36 = 2r^2(1 - 1/9)$, which yields $r = 9/2$ since $r > 0$. Hence the diameter of the semicircle is 9.

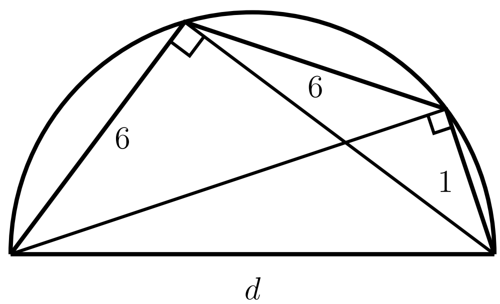
Option 3 (Ptolemy's theorem, straightforward)

The following solution uses a result in Euclidean geometry known as Ptolemy's theorem:

Theorem (Ptolemy) Let $ABCD$ be a cyclic quadrilateral, i.e., a quadrilateral whose vertices lie on a circle. We have $AB \cdot CD + BC \cdot AD = AC \cdot BD$.

That is, the product of the diagonals of a cyclic quadrilateral is equal to the sum of the pairwise products of its opposite sides.

Note that the chords and the diameter give us a cyclic quadrilateral with sides $1, 6, 6, d$, where d is the diameter of the semicircle. Since every triangle inscribed in a semicircle is a right triangle, each diagonal is the leg of a right triangle whose hypotenuse is the diameter, and whose other leg is another side of the quadrilateral. By the Pythagorean theorem, we have that the two diagonals are $\sqrt{d^2 - 36}$ and $\sqrt{d^2 - 1}$.



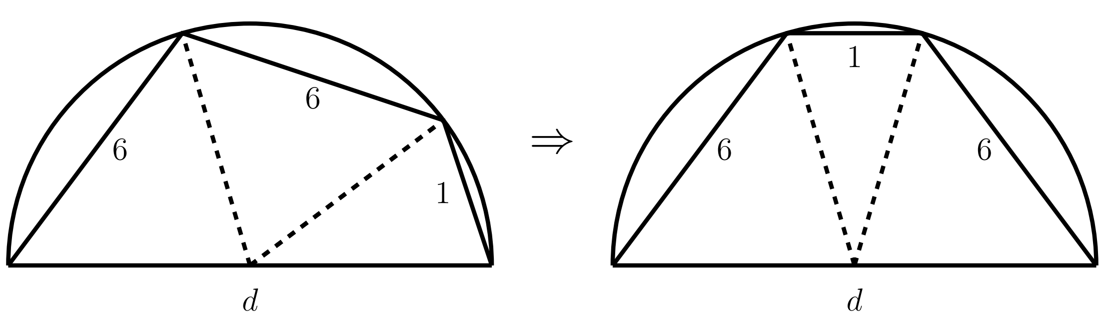
Now that we have the lengths of all sides of the cyclic quadrilateral as well as its diagonals, we can use Ptolemy's theorem. We get

$$\begin{aligned} 6 \cdot 1 + 6 \cdot d &= \sqrt{d^2 - 36} \cdot \sqrt{d^2 - 1} \\ (6 + 6d)^2 &= (d^2 - 36)(d^2 - 1) \\ 36d^2 + 72d + 36 &= d^4 - 37d^2 + 36 \\ d^4 - 73d^2 - 72d &= 0 \\ d(d-9)(d+1)(d+8) &= 0. \end{aligned}$$

Since $d > 0$, we must have $d = 9$.

Option 4 (Ptolemy with some cleverness)

The above calculation with Ptolemy's theorem can be simplified with some cleverness. Note that we can cut the semicircle into three arcs, and rearrange them, as follows:



In effect, the original quadrilateral has also been rearranged. Now, for this new quadrilateral, each diagonal has length $\sqrt{d^2 - 36}$. Ptolemy's theorem then gives us

$$\begin{aligned} 6 \cdot 6 + 1 \cdot d &= (\sqrt{d^2 - 36})^2 \\ d^2 - d - 72 &= 0 \\ (d-9)(d+8) &= 0. \end{aligned}$$

As before, since $d > 0$, we must have $d = 9$.

Problem 17. Let f be a real function satisfying the identity

$$f(x - 2023f(y)) = 1 - x - y$$

for all real numbers x and y. What is $f(-1)$?

(A) $\frac{2025}{2024}$ (B) $\frac{2024}{2023}$ (C) $\frac{2023}{2022}$ (D) $-\frac{2023}{2022}$ (E) 1 (F)$^{\heartsuit}$ There is no such function

Answer. There is no such function.[2]

Solution. Let $x = 2023f(y) + z$. The above equation becomes

$$f(z) = 1 - z - 2023f(y) - y$$

or

$$f(z) + z = 1 - y - 2023f(y).$$

Since the equality holds for all z, y there is a constant C such that $f(z) = z - C$ and $f(y) = \frac{1-y-C}{2023}$ however those identities cannot hold simultaneously as e.g. they represent lines with different slopes.

Problem 18. In a right triangle with sides of length a, b and c, we can compute the ratio

$$\rho = \frac{a+b+c}{c}$$

where c is the longest side. Which of these numbers is a possible value for ρ?

(A) 1.5 (B) 1.9 (C)$^{\heartsuit}$ 2.3 (D) 2.8 (E) 3.2

Answer. 2.3

Solution.

Option 1

Firstly, $c < a + b \implies 2c < a + b + c \implies 2 < \dfrac{a+b+c}{c}$.
Moreover, $(a-b)^2 \geq 0 \implies a^2 + b^2 \geq 2ab \implies 2(a^2+b^2) \geq (a+b)^2 \implies \sqrt{2}c \geq$

[2] The problem was removed as the initial answer was erroneously recorded as $\frac{2025}{2024}$.

$a+b \implies \sqrt{2}c+c \geq a+b+c \implies \sqrt{2}+1 \geq \dfrac{a+b+c}{c}$. Therefore, $1+\sqrt{2} \geq \dfrac{a+b+c}{c} \geq 2$ As $\sqrt{2} \approx 1.41$ we can conclude.

Option 2

The hypotenuse of a right triangle is relatively maximal when the two other sides have the same length (See Problem 11) and is relatively minimal when the triangle is degenerate, i.e. when one of the other sides is 0. In the former situation, $\rho = \frac{a+a+a\sqrt{2}}{a\sqrt{2}} = 1+\sqrt{2}$ and in the latter case $\rho = \frac{a+0+a}{a} = 2$. In the other scenarios, ρ varies between those bounds. We conclude in the same way.

Problem 19. Let $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ where $i = \sqrt{-1}$. Evaluate the following sum:

$$\sum_{a,b,c=1}^{30} z^{a^2b+bc^2-169b}.$$

(A)$^{\heartsuit}$ 12000 (B) $300-600i$ (C) $9000i$ (D) 9000 (E) $6000+3000i$

Answer. x

Solution. The important property of the complex number $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is that $z^3 = 1$, and that $z+z^2+z^3 = 0$. Note that for any integer x which is not divisible by 3, $y = z^x$ has the same property. If x is divisible by 3, on the other hand, $y+y^2+y^3 = 1+1+1 = 3$. One can extend this idea to a sum from 1 to 30,

$$\sum_{k=1}^{30} z^{kx} = \begin{cases} 0 & 3 \not| x \\ 30 & 3|x \end{cases}$$

We can take advantage of this by summing first in b, since the exponent can be factored as $b(a^2+c^2-169)$. We see that

$$\sum_{b=1}^{30} z^{b(a^2+c^2-169)} = \begin{cases} 0 & 3 \not|(a^2+c^2-1) \\ 30 & 3|(a^2+c^2-1) \end{cases}$$

Therefore, if there are N pairs (a,b) with $a,b \in \{1,...,30\}$ satisfying $a^2+c^2 \equiv 1 \pmod 3$, we find that $S = 30N$. To find N, note that $a^2+c^2 \equiv 1 \pmod 3$ if and only if exactly one of a and b is divisible by 3. Thus, $N = 2 \cdot 10 \cdot 20 = 400$, and so $S = 12000$.

Problem 20. . Let x,y be positive integers less than 100. How many (x,y) pairs satisfies the following equation?

$$2023x^2 = 2025y - 4$$

(A)$^\heartsuit$ 0 (B) 1 (C) 4 (D) 100 (E) 400

Answer. 0

Solution. The RHS has the remainder 1 on dividing by 5. Moreover, any integer has remainder $0, 1, 2, 3, 4$ on dividing by 5 so any integer square has remainder $0 = 0^2, 1 = 1^2 = 4^2$), or $4 = 2^2 = 3^2$) after division by 5. Then, any integer square times 2023 will have remainder $0 = 0 \times 3, 3 = 1 \times 3$, or $2 = 4 \times 3$. Therefore there is no integer pair that satisfies the equation as LHS has remainder $0, 2$, or 3 while the RHS has remainder 1 after division by 5.

Problem 21. Consider the list of all integers $n, m > 0$ such that

$$1! + 2! + 3! + \cdots + m! = n^2.$$

What is the product of all the elements in that list?

(A) 1 (B)$^\heartsuit$ 9 (C) 135 (D) 225 (E) 2025

Answer. 9

Solution. When working modulo 5, n^2 can only take 3 distinct values : $0^2 = 0$, $1^2 = 4^2 = 1$ and $2^2 = 3^2 = 4$. When $m \geq 5$, the left hand side of the equation reduces to $1! + 2! + 3! + 4! = 33 = 3$ modulo 5 and hence can never be a perfect square. For small values of m, one readily checks that the only possible answers are $1! = 1^2$ and $1! + 2! + 3! = 3^2$. In other words, the only solutions are $n = m = 1$ and $n = m = 3$ whose product is $1 \times 1 \times 3 \times 3 = 9$.

Problem 22. . How many pairs of integers (m, n), where $m, n \in 1, 2, 3, \ldots$ are such that $n^3 + m^3$ divides $n^2 + 6nm + m^2$?

(A) 1 (B) 2 (C) 3 (D) 4 (E)$^\heartsuit$ 5

Answer. 5

Solution. Note that if a, b are positive integers such that a divides b, then $a \leq b$. Thus, we have

$$\begin{aligned} m^3 + n^3 &\leq m^2 + 6mn + n^2 \\ m^3 + n^3 - m^2 + mn - n^2 &\leq 7mn \\ (m + n - 1)(m^2 - mn + n^2) &\leq 7mn \end{aligned}$$

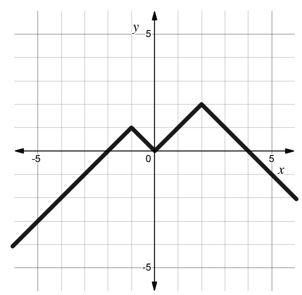
Now, note that $(m - n)^2 \geq 0$, and adding mn to both sides yields $m^2 - mn + n^2 \geq mn$ for all m, n; thus, we have $(m + n - 1)mn \leq 7mn$ and so $m + n - 1 \leq 7$, i.e. $m + n \leq 8$.

Assume for now, without loss of generality, that $m \leq n$; we will then have $n \leq 4$.

- For $n = 1$, we can only have $m = 1$, which indeed works.
- For $n = 2$, we have $m = 1$ or $m = 2$. We have that $m = 2$ works.
- For $n = 3$, we have that $m = 1$ works.
- Finally, for $n = 4$, we have that $m = 4$ works.

Accounting now for the pairs with $m > n$, we get a total of 5 solutions.

Problem 23. The continuous piecewise linear function f is depicted below. The graph has 3 corners. How many corners does $(f \circ f)(x) = f(f(x))$ have?



(A) 3 (B) 4 (C) 5 (D) 6 (E)$^\heartsuit$ 7

Answer. 7

Solution. The singular points of f lie above -1, 0 and 2. One can see from the graph that the preimage of -1 has two points : $f^{-1}(-1) = \{3, 5\}$; and that $f^{-1}(0) = \{-2, 0, 4\}$ and $f^{-1}(2) = \{2\}$. Above these 6 values we will have corners. To these points, we must also add the point above -1 which remains angular as around $1 = f(-1)$, f is linear.

Problem 24. . Consider the set $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. We want to cover S with 5 closed intervals of equal length. What is the minimal length of each individual interval?

(A)$^\heartsuit$ $\frac{1}{10}$ (B) $\frac{3}{25}$ (C) $\frac{1}{8}$ (D) $\frac{1}{6}$ (E) $\frac{1}{5}$

Answer. $\frac{1}{10}$

Solution. The minimal length is $\frac{1}{10}$. Let's split S into the following disjoint sets: $S_1 = \{1\}$, $S_2 = \{\frac{1}{2}\}$, $S_3 = \{\frac{1}{3}, \frac{1}{4}\}$, $S_4 = \{\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\}$ and $S_5 = \{\frac{1}{10}, \frac{1}{11}, \ldots\}$. These sets can all be covered by intervals of length $\frac{1}{10}$ – this is obvious for S_1, S_2 and S_5 and for S_3 (resp. S_4), note that $\frac{1}{3} - \frac{1}{4} \leq \frac{1}{10}$ (resp. $\frac{1}{5} - \frac{1}{9} \leq \frac{1}{10}$).
Had we chosen an interval with smaller length $\ell < \frac{1}{10}$, then we would have not been able to cover all elements of S. Clearly, S_1 and S_2 would still need two intervals to be

covered. S_3 could still be covered but we could not include $\frac{1}{5}$ as $\frac{1}{3} - \frac{1}{5} > \frac{1}{10}$. Keeping S_5 to the right (in order to include the tail of S), we must place $\frac{1}{5}$ and $\frac{1}{10}$ in S_4, but $\frac{1}{5} - \frac{1}{10} = \frac{1}{10} > \ell$, a contradiction.

Problem 25. In the equation below, a, b, c, and d are base-10 digits. Moreover, assume that neither a nor c equals 0. What is cd if

$$abcd = (ab)^2 + (cd)^2?$$

Here, $abcd$ is a 4-digit number, and ab and cd are 2-digit numbers.

(A) 30 (B) 31 (C) 32 (D)$^\heartsuit$ 33 (E) 34

Answer. 33

Solution.
Option 1 Let $m := ab$, $n := cd$. Then we have

$$\begin{aligned} 100m + n &= m^2 + n^2 \\ m^2 - 100m + n^2 - n &= 0 \\ 4m^2 - 400m + 4n^2 - 4n &= 0 \\ (2m - 100)^2 + (2n - 1)^2 &= 10001. \end{aligned}$$

Let $x := 2m - 100, y := 2n - 1$. We know that x is even and y is odd; we want to find ordered pairs of integers (x, y) with x even, y odd such that $x^2 + y^2 = 10001$. Since there are only finitely many possible values of x, y, we may use trial and error. We have the following solutions in (x, y): $(\pm 100, \pm 1), (\pm 76, \pm 65)$. As it turns out, only the solutions $(\pm 76, 65)$ give allowable values of m, n. We have, if $(x, y) = (-76, 65)$, $(m, n) = (12, 33)$; if $(x, y) = (76, 65)$, we have $(m, n) = (88, 33)$. In both cases, $n = 33$.

Option 2 Using complex numbers (specifically, the Gaussian integers) and some factorization tricks, there is another way to find the above solutions that requires a bit less guesswork.

Note that $10001 = 73 \cdot 137$. To see this, we test squares just above $10001 > 100^2$ and see which ones differ from 10001 by a perfect square. We then use the difference of squares factorization to factor. We have:

$$\begin{aligned} &101^2 - 10001 = 200 && \text{not a square} \\ &102^2 - 10001 = 403 && \text{not a square} \\ &103^2 - 10001 = 608 && \text{not a square} \\ &104^2 - 10001 = 815 && \text{not a square} \\ &105^2 - 10001 = 1024 = 32^2 && \end{aligned}$$

From the above, we have $10001 = 105^2 - 32^2 = (105 - 32)(105 + 32) = 73 \cdot 137$.

This factorization turns out to be useful. Consider the norm $\mathrm{N} : \mathbb{C} \to \mathbb{R}_{\geq 0}$ given by $\mathrm{N}(a+bi) = a^2 + b^2$ for all real numbers a, b. It is not hard to show that the norm is multiplicative, i.e., if $\alpha, \beta \in \mathbb{C}$, we have $\mathrm{N}(\alpha\beta) = \mathrm{N}(\alpha)\mathrm{N}(\beta)$. This means that if there exist α, β with $\mathrm{N}(\alpha) = 73$ and $\mathrm{N}(\beta) = 137$, then $\mathrm{N}(\alpha\beta) = 73 \cdot 137 = 10001$. Moreover, if we write $\alpha = a_1 + b_1 i$, $\beta = a_2 + b_2 i$ with $a_1, a_2, b_1, b_2 \in \mathbb{Z}$, then we can write $\alpha\beta = a_3 + b_3 i$ where $a_3, b_3 \in \mathbb{Z}$ as well. This gives us a way of finding integers $x = a_3, y = b_3$ satisfying $x^2 + y^2 = 10001$.

In fact, we can say more. The Gaussian integers $\mathbb{Z}[i] := \{a + bi : a, b \in \mathbb{Z}\}$, i.e., the set of all numbers of the form $a + bi$, where a and b are integers, satisfy a unique factorization property just like that of the usual integers. That is, every Gaussian integer $\alpha \neq 0$ has a prime factorization

$$\alpha = \mu \pi_1^{r_1} \dots \pi_k^{r_k}$$

for some unit μ, some primes $\pi_1, \dots, \pi_k$, and some positive (usual) integers $r_1, \dots, r_k$.

In this setting, the primes are those of the form $a+bi$ wherein $a^2+b^2 = p$ for some prime p, as well as the usual rational primes $p \equiv 3 \pmod 4$, up to multiplication by some unit. The units here are the powers of i, i.e., ± 1 and $\pm i$.

This prime factorization is unique, up to reordering and multiplication by units. That is, maybe we have two ways to write α as a product of units and primes. The total number of primes, up to multiplicity is always the same; moreover, the actual list of primes should be the same, except maybe we have π_j in the first list and $\mu\pi_j$ in the second list for some unit μ.

One useful, if nonstandard way to formulate the above: there is a unique, up to order, way to write

$$\alpha = \mu \alpha_1^{r_1} \dots \alpha_k^{r_k}$$

where the primes $\pi_1, \dots, \pi_k$ are distinct, such that if $\pi_j = a_j + b_j i$ we have $a_j > 0$, $b_j \geq 0$ for all $1 \leq j \leq k$.

The above unique factorization property tells us that, in fact, if $\mathrm{N}(a+bi) = 10001$, we must be able to write $a + bi = \pi_1\pi_2$ for some primes $\pi_1 = a_1 + b_1 i, \pi_2 = a_2 + b_2 i$ with $\mathrm{N}(\pi_1) = 73$ and $\mathrm{N}(\pi_2) = 137$. The only way to write 73 as a sum of two squares, up to order and signs, is $73 = 8^2 + 3^2$; similarly, we only have $137 = 11^2 + 4^2$.

Moreover, while finding a way to write an integer $n = a^2 + b^2$ as a sum of two squares is equivalent to finding a Gaussian integer α with $\mathrm{N}(\alpha) = n$, we have that, for any unit μ, α and $\mu\alpha$ do not give a fundamentally different pair a, b. This is because if $\alpha = a + bi$, we have $i\alpha = -b + ai$, and every unit is a power of i, so we always get the same a, b up to order and signs. It thus suffices to test combinations of π_1, π_2 with $a_1, a_2 > 0$ and $b_1, b_2 \geq 0$. We have

$$\begin{aligned}
(3 + 8i)(4 + 11i) &= -76 + 65i \\
(3 + 8i)(11 + 4i) &= 1 + 100i \\
(8 + 3i)(4 + 11i) &= -1 + 100i \\
(8 + 3i)(11 + 4i) &= 76 + 65i
\end{aligned}$$

This tells us that the only ways to write 10001 as the sum of two squares are $10001 = 100^2 + 1^2 = 76^2 + 65^2$, up to sign and order.

$\diamond \diamond \diamond$

Authors. Written and edited by Paco Adajar, Jimmy Dillies, Mo Hendon, Gary Iliev, Tekin Karadag, Brian McDonald, Paul Pollack and Casia Siegel.