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Written test, 25 problems / 90 minutes

October 22, 2022

WITH SOLUTIONS

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. Suppose a, b, c, and d are nonzero digits 1, 2, …, 9 and

$$ab^c - d = 2022.$$

What is $a + b + c + d$? Note: Here ab is a two digit number, not a product.

(A) 11 (B) 12 (C) 13 (D)$^\heartsuit$ 14 (E) 15

Solution. Since ab is a two digit number, $ab^1 < 100$ and $ab^4 > 10000$, so c must be 2 or 3. The only two digit numbers whose cubes are less than 2500 are $10^3 = 1000$, $11^3 = 1331$, $12^3 = 1728$, and $13^3 = 2197$. None of these are within 9 of 2022, so we must have $c = 2$.

The two digit number whose square is closest to 2022 is 45; indeed $45^2 = 2025$, and

$$2022 = 45^2 - 3.$$

Therefore $a = 4$, $b = 5$, $c = 2$, and $d = 3$. Thus $a + b + c + d = 14$.

Problem 2. Let a and b be nonnegative integers such that $ab < a^b < a+b$. To which of the following is $(a + b)^2$ equivalent.

(A)$^\heartsuit$ a^2 (B) b^2 (C) ab (D) $4ab$ (E) $a + b$

Solution. Note that $ab + 1 < a + b$, so $ab - a - b + 1 < 0$, but the left-hand side can be factored as $(a - 1)(b - 1)$, so we find that either a or b must be 0. If $a = 0$, the inequality $ab < a^b$ fails, so $b = 0$. When $b = 0$ the last inequality forces $a > 1$ and indeed any choice of such an a is valid. Hence $(a + b)^2 = (a + 0)^2 = a^2$.

Problem 3. You can write

$$x^{1022} = (x^2 - 1)f(x) + Ax + B$$

for some polynomial $f(x)$, where A and B are real numbers. What is $A^2 + B^2$?

(A) 0 (B) $\frac{1}{2}$ (C)$^\heartsuit$ 1 (D) 2 (E) $\frac{5}{2}$

Solution. First Approach: Substituting $x = 1$ gives us $1 = A + B$; substituting $x = -1$ gives us $1 = -A + B$. Solving this system of equations readily gives us $A = 0$, $B = 1$ and so $A^2 + B^2 = 1$.

Second Approach: More explicitly, recalling that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$, we write $x^{1022} - 1 = (x^2)^{511} - 1 = (x^2 - 1)(x^{1020} + x^{1018} + \cdots + x^2 + 1)$ and so $x^{1022} = (x^2 - 1)(x^{1020} + x^{1018} + \cdots + x^2 + 1) + 1$. This gives us $A = 0, B = 1$ as in Solution 1.

Problem 4. Let $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + x^n$ be the monic polynomial of the lowest degree vanishing exactly at the positive divisors of 16 (i.e. $p(m) = 0$ if and only if $m \geq 0$ and m divides 16). What is $p(0)$?

(A) -31 (B) 31 (C) 256 (D)$^\heartsuit$ -1024 (E) 1024

Solution. A polynomial $p(x)$ will vanish at a if and only if it can be factored as $p(x) = (x - a)q(x)$. Since the divisors of 16 are $1, 2, 4, 8$ and 16, and $p(x)$ is of minimal degree with 1 as leading coefficient, it can written as

$$p(x) = (x - 1)(x - 2)(x - 4)(x - 8)(x - 16).$$

Evaluating the above expression at 0, we get $(-1)^5 1 \cdot 2 \cdot 4 \cdot 8 \cdot 16$ or

$$-1024.$$

Problem 5. Suppose you have 100 positive integers with the property that

$$a_1 < a_2 < \cdots < a_{99} < A < a_{100},$$

where A is the average of the 100 numbers $a_1, \ldots, a_{100}$. What is the smallest that a_{100} can be?

(A) 100 (B) 4950 (C)$^\heartsuit$ 4951 (D) 5050 (E) This is not possible

Solution. Assume a_{100} is minimized. Under this assumption $a_1 = 1$ is forced, otherwise we could decrease every a_k by 1 without changing the relative position of the average. Also note that a_{k-1} and a_k must be consecutive for $k \leq 99$, else we could increase a_{k-1} by 1 and decrease a_{100} by 1 without changing the average A, contradicting the minimality of a_{100}. Hence $a_1 = 1, \ldots, a_{99} = 99$. Then the average of the 100 numbers is

$$\frac{1 + 2 + \cdots + 99 + a_{100}}{100} = \frac{4950 + a_{100}}{100}.$$

We want this to be bigger than $a_{99} = 99$:

$$\frac{4950 + a_{100}}{100} > 99 \iff 4950 + a_{100} > 9900 \iff a_{100} > 4950,$$

so the smallest that a_{100} can be is 4951.

Problem 6. The average of n numbers $a_1, a_2, \ldots, a_n$ is 10. If each a_k is increased by k, the n resulting numbers have an average of 20. What is n?

(A) 4 (B) 10 (C)$^\heartsuit$ 19 (D) 20 (E) There is not a unique solution.

Solution. The average of $a_1, a_2, \ldots, a_n$ is 10, that is

$$10 = \frac{1}{n} \sum_{k=1}^{n} a_k.$$

The average of the increased numbers is 20;

$$\begin{aligned} 20 &= \frac{1}{n} \sum_{k=1}^{n} [a_k + k] \\ &= \frac{1}{n} \sum_{k=1}^{n} a_k + \frac{1}{n} \sum_{k=1}^{n} k \\ &= 10 + \frac{1}{n} \frac{n(n+1)}{2} \\ &= 10 + \frac{n+1}{2} \end{aligned}$$

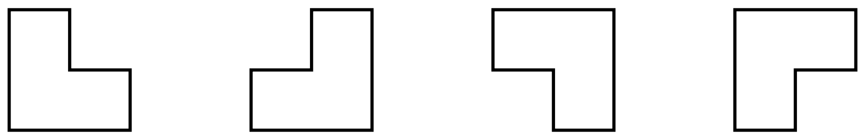
Hence $n = 19$.

Problem 7. We know that there are $8! = 40320$ permutations of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. How many of those permutations map the small numbers $\{1, 2, 3, 4\}$ to the large numbers $\{5, 6, 7, 8\}$?

(A) 24 (B)$^\heartsuit$ 576 (C) 2520 (D) 10080 (E) 20160

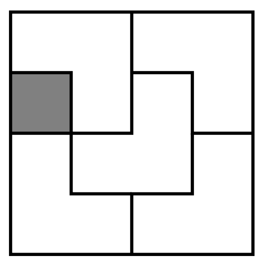
Solution. If $\sigma(x)$ is such a permutation, then there are four options for $\sigma(1)$, three remaining options for $\sigma(2)$, two remaining options for $\sigma(3)$, and only one remaining option for $\sigma(4)$. There are the same options for $\sigma(5)$, $\sigma(6)$, $\sigma(7)$, and $\sigma(8)$, for a total of $(4!)^2 = 576$ options.

Problem 8. An *L-tromino* is a shape made by gluing three unit squares into the shape of an L. Take a 4×4 square board, consisting of 16 unit squares, and choose a unit square. It is always possible to cover the entire board except for the chosen square with L-trominos, rotating them if necessary. In such a tiling, each L-tromino is in one of four different orientations:

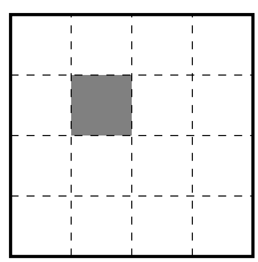


Orientation 1 Orientation 2 Orientation 3 Orientation 4

For a tiling of the board minus a square, define a_1 be the number of L-trominos in Orientation 1 (as defined above), and similarly define a_2, a_3, a_4. For example, in the tiling below, we have $a_1 = 1$, $a_2 = a_3 = 2$, $a_4 = 0$.



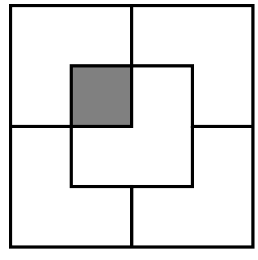
Consider now a tiling of the board minus the shaded square shown below:



Which of the following is the largest?

(A) a_1 (B)$^\heartsuit$ a_2 (C) a_3 (D) a_4 (E) Cannot be determined

Solution. The following is a tiling for the board above. In this tiling, we have $a_1 = a_3 = a_4 = 1$, $a_2 = 2$.



This tiling is, in fact, the only possible tiling. To deduce this, first we note that in order to cover the square in the upper-left corner, but leave the shaded square uncovered, we can only have an L-tromino in Orientation 4 at the upper-left corner. Then, to cover the upper-right corner, we can only have an L-tromino in Orientation 2, 3, or 4. An L-tromino in Orientation 2 would leave a square on the upper edge of the board uncovered. An L-tromino in Orientation 4 would force an L-tromino in Orientation 1 to cover the "missing" square from that first L-tromino on the right edge; this then leaves two squares in the lower-right corner that cannot be covered. Thus, we must in fact have an L-tromino in Orientation 3 in the upper-right corner. A similar argument shows that we must have an L-tromino in Orientation 2 in the lower-right corner, then an L-tromino in Orientation 1 in the lower-left corner. Finally, we have three other uncovered squares, and these are covered exactly by an L-tromino in Orientation 2. This gives us the exact tiling above.

Problem 9. Find a digit y, that is $y \in \{0, 1, \ldots, 9\}$, so that for *every* digit x, the 10 digit number

$$1022xy2022$$

is not divisible by 11.

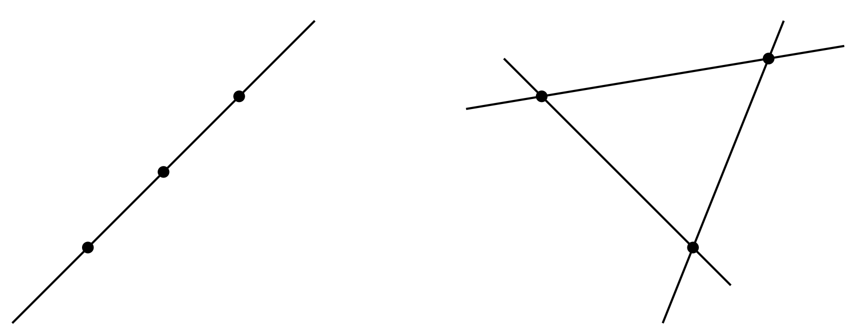
(A) 0 (B)$^\heartsuit$ 2 (C) 4 (D) 6 (E) 8

Solution. $1022xy2022$ is divisible by 11 if and only if

$$1 - 0 + 2 - 2 + x - y + 2 - 0 + 2 - 2 = 3 + x - y$$

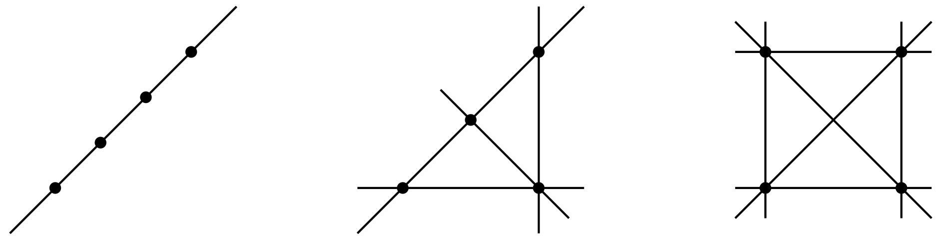
is divisible by 11. Since x and y are small, this means either $3+x-y = 0$ or $3+x-y = 11$. The first says that $x = y - 3$, which for $y \geq 3$ has solutions $x \in \{0, \ldots, 9\}$. The second says $x = 8 + y$, which for $y \leq 1$ has solutions in $x \in \{0, \ldots, 9\}$. So we take $y = 2$: $1022x22022$ is not divisible by 11 for any digit x.

Problem 10. We all know two distinct points in the plane determine one line, while three distinct points determine either one or three lines; depending on whether the points are colinear:



So we say there are 2 different configurations of 3 points.

Similarly, there are 3 different configurations of 4 points:

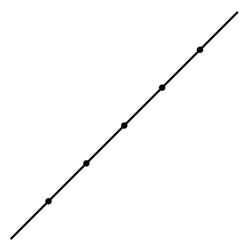


How many different configurations of 5 points are there?

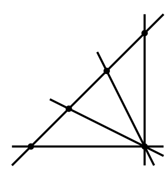
(A) 4 (B)$^\heartsuit$ 5 (C) 6 (D) 8 (E) 9

Solution. We can organize our count based on the largest number of points on one line:

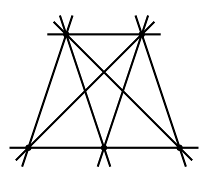
Max 5 points on a line determine a single line:

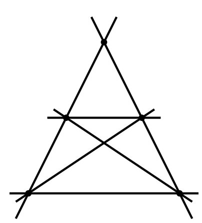


Max 4 points on a line determine 5 lines:



Max 3 points on a line determine either 8 or 6 lines:





Max 2 points on a line determine 10 lines:

Problem 11. How many rectangles are there in a 2022×2022 chessboard? Yes, one of the listed options is the correct answer!

(A) 4183059834002 (B) 4183059834003 (C) 4183059834007 (D) 4183059834008 (E)$^\heartsuit$ 4183059834009

Solution. If one assumes the correct answer is among those listed, this is a quick problem after the following two observations:

1. The number of rectangles in an $n \times n$ square is a perfect square.

2. The rightmost digit of a perfect square can only be 0, 1, 4, 5, 6, or 9.

The second claim can be checked immediately by looking at all possible squares modulo 10. The first claim can be seen by observing that a rectangle is uniquely determined by a choice of two horizontal boundary lines and two vertical boundary lines. If there are x ways to choose the horizontal lines, then there are also x ways to choose the vertical lines, so there are a total of x^2 ways of choosing the rectangle.

In fact, here there are 2023 places to put a vertical boundary line (noting the edges of the board are valid boundary lines), so we get

$$x^2 = \binom{2023}{2}^2 = 2023^2 \cdot 1011^2 = 4183059834009.$$

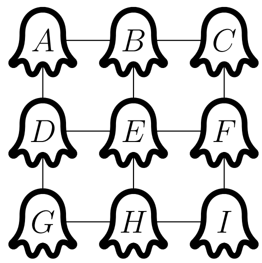
Problem 12. Assume (x, y, z) is a Pythagorean triple, that is $x^2 + y^2 = z^2$. Assume x and z are successive odd numbers and x is not divisible by 3. Which of the following is guaranteed to be a divisor of y?

(A) 9 (B) 10 (C)$^\heartsuit$ 12 (D) 15 (E) 21

Solution. We can write $x^2+y^2 = (x+2)^2 = x^2+4x+4$, which reduces to $y^2 = 4(x+1)$. Since x is odd, $(x + 1)$ must be even, so y must in fact be divisible by 4. Moreover, if we divide any whole square by 3, the remainder is 0 or 1. If both x and y are not divisible by 3, the remainder for z^2 will be 2 which is a contradiction. Therefore, y must be divisible by 3. Hence, y is always divisible by 12.

The following description will be used for the next three problems. Beware: they're squary! Define a *spooky square* with *spooky sum* S to be an arrangement of the numbers 1 through 9 in the nine spaces below such that each of the four 2×2 subsquares adds to the same number S. That is,

$$S = A + B + D + E = B + C + E + F = D + E + G + H = E + F + H + I.$$

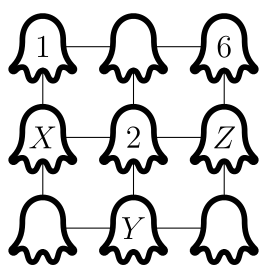


Problem 13. Which of the following is equivalent to $C + G - A$ for all spooky squares?

(A) A (B) $3E - F - H$ (C) $B + D - E$ (D) $F + H - E$ (E)$^\heartsuit$ I

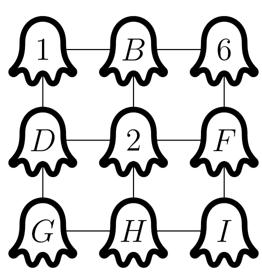
Solution. Using the two squares on the left we have $A+B+D+E = D+E+G+H$, so we find $A + B = G + H$. Similarly using the two squares on the right, we have $B + C = H + I$. Taking the difference in these equations $A - C = G - I$, so $I = C + G - A$.

Problem 14. Complete the spooky square below. What is $X + Y + Z$?

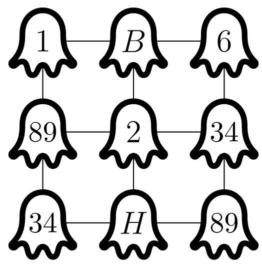


(A) 16 (B) 17 (C)$^\heartsuit$ 18 (D) 19 (E) 20

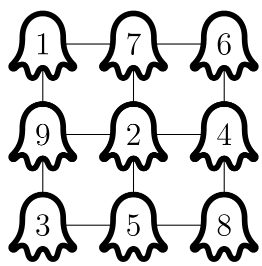
Solution.



Note in the spooky square above we have $1 + D = 6 + F$ and $D + G = F + I$. Hence $5 = D - F = I - G$. There are only two pairs of digits remaining which have difference 5: They are $\{3, 8\}$ and $\{4, 9\}$. Right now we don't know exactly what is what, but from the equation, we know the larger pair of digits is $\{D, I\} = \{8, 9\}$ and the smaller pair of digits is $\{F, G\} = \{3, 4\}$. Filling in the options, we get:



This leaves $\{B, H\} = \{5, 7\}$. Now we can use the symmetric fact as the one we started with, namely that $B - H = G - 1$. However $|B - H| = |7 - 5| = 2$, and as G is restricted to 3 or 4, we must have $G = 3$ so that $|G - 1| = 2$. This fixes the entire spooky square and the full solution is:



Hence $X = 9$, $Y = 5$, and $Z = 4$, so $X + Y + Z = 18$.

Problem 15. There are 376 distinct spooky squares. What is the average of all 376 spooky square sums?

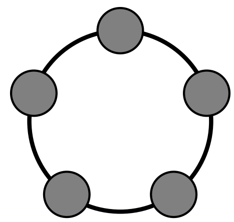
(A) 17 (B) 18 (C) 19 (D)$^\heartsuit$ 20 (E) 21

Solution. Every spooky square has a dual spooky square in which every digit x is replaced with $10-x$. Note that if a spooky square has spooky sum $S = A+B+D+E$, then its dual has spooky sum

$$10 - A + 10 - B + 10 - D + 10 - E = 40 - S.$$

Hence the average of the spooky sum of a spooky square and the spooky sum of its dual spooky square is $\frac{1}{2}(S + 40 - S) = 20$. Since every spooky square has a spooky dual, the average of spooky sums over all spooky squares is also 20.

Problem 16. Five lamps are on a circular circuit, as shown below. Manually toggling a lamp, i.e., switching it from off to on or vice versa, automatically toggles the two lamps adjacent to it.

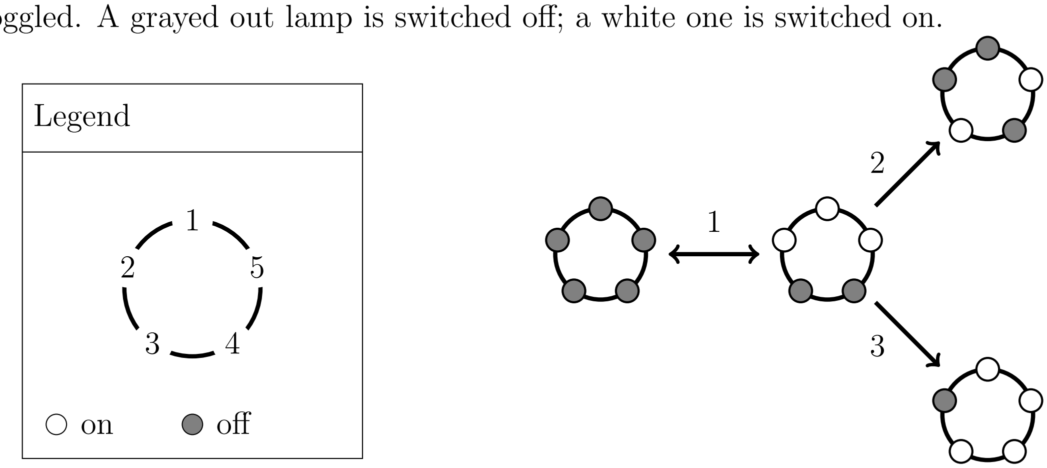


Initially, all lamps are switched off. What is the least number of moves (i.e., manual toggles) required to have them all switched on at the same time?

(A) 3 (B) 4 (C)$^\heartsuit$ 5 (D) 6 (E) 7

Solution. First Approach: It is possible to switch all lamps on in five moves; toggling each lamp once works, and order doesn't matter. We show that we can do no better.

First note that to get every lamp switched on, we need to toggle each lamp an odd number of times, and thus the total number of times all lamps are toggled must be odd. Since an odd number of lamps is toggled for each move, the total number of moves then has to be odd. It remains to show that it is impossible to do it in three moves. If it were possible to switch on all lamps in three moves, it is then possible to switch on exactly two adjacent lamps after two turns. However, this is not possible. After the first turn, three adjacent lamps are switched on. Depending on whether you switch on the middle lamp (same as you toggled on the first turn), one of the other two lamps, or one of the two lamps left switched on, the lamps switched on will either be none, two non-adjacent lamps, or four lamps, as shown in the figure below. For reference, we are numbering the lamp on top as 1, and counting 2, 3, 4, 5 counterclockwise. The number beside each arrow indicates the number of the lamp toggled. A grayed out lamp is switched off; a white one is switched on.



Second Approach Another way to think about this question is as a system of equations over the finite field $\mathbb{F}_2$. In this finite field, the only elements are 0 and 1, and addition (+) and multiplication ($\cdot$) are defined as with the rules for even and odd, with 0 being even and 1 being odd: $0+0 = 1+1 = 0$, $0+1 = 1+0 = 1$, $0\cdot0 = 0\cdot1 = 1\cdot0 = 0$, $1 \cdot 1 = 1$. Notice that in this field, addition and subtraction are the same, and that $2 = 0$.

Number the lamps 1, 2, 3, 4, 5, as previously. Let n_1 be the number of times Lamp 1 was manually toggled, and similarly define n_2, n_3, n_4, n_5. Note that the order in which the lamps were toggled doesn't matter, and that toggling a lamp twice has net zero effect on the whole configuration; thus, we can take n_1, n_2, n_3, n_4, n_5 to be 0 or 1, and view them as elements of $\mathbb{F}_2$.

The total number of times Lamp 1 is toggled, manually or automatically, is given by $n_5 + n_1 + n_2$, and we need this to equal 1 so that Lamp 1 is switched on. Similar equations hold for the other four lamps. Thus, we have the system of equations

$$\begin{aligned}
n_5 + n_1 + n_2 &= 1 \\
n_1 + n_2 + n_3 &= 1 \\
n_2 + n_3 + n_4 &= 1 \\
n_3 + n_4 + n_5 &= 1 \\
n_4 + n_5 + n_1 &= 1.
\end{aligned}$$

There are several ways to solve the above system. For instance, by adding the first, third, and fourth equations together, we have

$$(n_5 + n_1 + n_2) + (n_2 + n_3 + n_4) + (n_3 + n_4 + n_5) = 1 + 1 + 1$$

which simplifies to $n_1 = 1$. By rotational symmetry or similarly solving, we must have that $n_2 = n_3 = n_4 = n_5 = 1$ as well. Thus, viewing n_1, n_2, n_3, n_4, n_5 as integers as originally, we get a total of at least five moves.

Problem 17. In how many ways can you write 100 as an unordered sum of 1's, 2's, and 3's. Unordered here means $1+2$ counts the same as $2+1$ and similarly for other rearrangements of sums.

(A) 520 (B)$^\heartsuit$ 884 (C) 948 (D) 1020 (E) 1326

Solution. First note that any such combination has between 0 and 33 copies of 3, and so the total number of combinations can be calculated by adding up the number of combinations of 1 and 2 that add up to $100 - 3n$, as n ranges between 0 and 33. We then consider two separate cases:

1. n is even: In this case, $100 - 3n$ can be written as $100 - 6k$, where k runs between 0 and 16. Each combination of 1 and 2 adding to this value is uniquely determined by the number of twos, which can range between 0 and 50-3k, giving a total of $51 - 3k$ possibilities, or a total sum of

$$\sum_{k=0}^{16} (51 - 3k)$$

for the even values of n.

2. n is odd: Here, $100-3n$ can be written as $97-6k$, again with k running between 0 and 16. Here, the number of twos ranges between 0 and $48-3k$, giving a total of $49-3k$ possibilities, or a total sum of

$$\sum_{k=0}^{16}(49-3k)$$

for the odd values of n.

Combining these two sums, we get that the total number of combinations is equal to

$$\sum_{k=0}^{16}(100-6k)=\sum_{k=0}^{16}100-6\sum_{k=0}^{16}k=1700-3(16^2+16)=1700-816=884.$$

Problem 18. The number S given by the infinite sum

$$S=\sum_{n=1}^{\infty}\frac{1}{10^n-1}=\frac{1}{9}+\frac{1}{99}+\frac{1}{999}+\dots$$

has decimal expansion

$$S=0.d_1d_2d_3\cdots=\sum_{k=1}^{\infty}d_k10^{-k}$$

where d_k is the kth decimal digit after the decimal point. Find d_{22}.

(A) 3 (B)$^\heartsuit$ 4 (C) 5 (D) 6 (E) 7

Solution. Note that, for each n, we have

$$\frac{1}{10^n-1}=\sum_{m=1}^{\infty}10^{-mn}$$

and so we have

$$S=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}10^{-mn}.$$

Now, for each k, we count the number of times 10^{-k} appears in this iterated sum. This is given by the number of ordered pairs of positive integers (m,n) with $mn=k$. Since each pair is in fact uniquely determined by m, it boils down to counting the number of divisors $\tau(k)$ of k. Thus,

$$S=\sum_{k=1}^{\infty}\tau(k)10^{-k}.$$

Another, less formal way to see the above: note that $1/9 = 0.11\ldots$, $1/99 = 0.0101\ldots$, $1/999 = 0.001001\ldots$ and so on and so forth. That is,

$$1/\underbrace{99\ldots9}_{n\text{ 9s}} = 1/(10^n - 1) = 0.\underbrace{0\ldots0}_{n-1\text{ 0s}}1\ldots.$$

Each term $1/(10^n-1)$ adds a 1 to every place after the decimal point that is a multiple of n. Thus, the number of times 1 is added to the kth decimal place is given by the number of divisors of k, which we have just called $\tau(k)$.

Comparing this to the original series suggests something like $d_k = \tau(k)$. Of course, this is not always going to be the case. For instance, we must have $d_k < 10$, but sometimes $\tau(k) \geq 10$. Also, even if $\tau(k) < 10$, it may be that "carrying-over" in succeeding digits adds one or more to the final digit value in the sum. This happens exactly when $\sum_{i=k+1}^{\infty} \tau(i)10^{-i} > 10^{-k}$. However, when $\tau(k) < 10$ and there is no carrying-over, we indeed have that $d_k = \tau(k)$.

Now, we have that $\tau(22) = 4$. To show that no carrying-over happens, it suffices to show that

$$\sum_{i=23}^{\infty} \tau(i)10^{-i} < 10^{-22}.$$

We can write

$$\begin{aligned}
\sum_{i=23}^{\infty} \tau(i)10^{-i} &= \tau(23)10^{-23} + \tau(24)10^{-24} + \sum_{i=25}^{\infty} \tau(i)10^{-i} \\
&= 2 \cdot 10^{-23} + 8 \cdot 10^{-24} + \sum_{k=1}^{\infty} \tau(k+24)10^{-(k+24)} \\
&= 2 \cdot 10^{-23} + 8 \cdot 10^{-24} + 10^{-24} \sum_{k=1}^{\infty} \tau(k+24)10^{-k}.
\end{aligned}$$

Note that $\tau(n) < n$ for all n; we thus have $\tau(k+24) \leq k+24 \leq 25k$ for all $k \geq 1$. Now, we have that

$$\begin{aligned}
\sum_{k=1}^{\infty} k10^{-k} &= \sum_{k=1}^{\infty} \sum_{i=1}^{k} 10^{-k} \\
&= \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} 10^{-k} \\
&= \sum_{i=1}^{\infty} \frac{1}{9} 10^{-i+1} \\
&= \frac{1}{9} \cdot \frac{10}{9} = \frac{10}{81}.
\end{aligned}$$

Informally: we may write the infinite sum $\sum_{k=1}^{\infty} 10^k$ as

$$\begin{array}{ccccccc}
10^{-1} & + & 10^{-2} & + & 10^{-3} & + & \ldots \\
 & + & 10^{-2} & + & 10^{-3} & + & \ldots \\
 & & & + & 10^{-3} & + & \ddots
\end{array}$$

Summing by column first, then adding all the column sums gives the original series; summing first by row instead gives us a geometric series of geometric series as in above.

(For those who know calculus, we may also obtain the above by differentiating the geometric series for $1/(1-x) = \sum_{k=0}^{\infty} x^k$ and multiplying by x, then substituting $x = 1/10$.)

This gives us that

$$\begin{aligned}\sum_{k=23}^{\infty} \tau(k)10^{-k} &\leq 2 \cdot 10^{-23} + 8 \cdot 10^{-24} + 10^{-24} \cdot 25 \sum_{k=1}^{\infty} k10^{-k} \\ &= 2 \cdot 10^{-23} + 8 \cdot 10^{-24} + 25 \cdot 10^{-24} \cdot \frac{10}{81} \\ &= 10^{-24}\left(28 + \frac{250}{81}\right) \\ &< 10^{-24}(100) = 10^{-22}\end{aligned}$$

as desired. That is, there is no carrying over, and so $d_{22} = \tau(22) = 4$.

Aside: In fact, Paul Erdős proved S is irrational in his 1948 paper *On arithmetical properties of Lambert series.*

Problem 19. The nonzero digits A, B, C, D are such that the two-digit number AB divides the two-digit number CD, and the four-digit number $ABCD$ is a perfect square. Find $A + B + C + D$.

(A) 9 (B) 12 (C) 16 (D)$^{\heartsuit}$ 18 (E) 25

Solution. A guess and check approach is available: Checking the small four-digit squares, we have $32^2 = 1024$, $33^2 = 1089$, $34^2 = 1156$, $35^2 = 1225$, $36^2 = 1296$. Indeed, for the last, 12 divides 96. Thus, we have $1 + 2 + 9 + 6 = 18$.

A rigorous approach to the problem is as follows: We can write $AB = n, CD = kn$ for some constant k; we then have $ABCD = (100 + k)n$. Note that k must be small; in fact, it must be a decimal digit as well.

Now, we recall that any positive integer m can be written uniquely as $m = d^2m'$ where m' is the squarefree part of m, that is, the largest squarefree (not divisible by any perfect square greater than 1) number dividing m. More specifically, m' is the product of all primes that appear an odd number of times in the prime factorization of m. Moreover, the product of two positive integers is a square if and only if they have the same squarefree part. As $ABCD = (100 + k)n$ is a square, we must have $100 + k = c^2n'$, $n = d^2n'$ for some positive integers c, d, n' with n' squarefree. Below is a table showing the factorization and squarefree part of $100 + k$ for $1 \leq k \leq 9$:

k	Prime factorization of $100+k$	Squarefree part (n')
1	$101 = 101$	101
2	$102 = 2 \cdot 3 \cdot 17$	102
3	$103 = 103$	103
4	$104 = 2^3 \cdot 13$	26
5	$105 = 3 \cdot 5 \cdot 7$	105
6	$106 = 2 \cdot 53$	106
7	$107 = 107$	107
8	$108 = 2^2 \cdot 3^3$	3
9	$109 = 109$	109

From the above table, we can see that $k = 4$ or $k = 8$; otherwise, $n' > 100$, and this cannot be because n' divides a two-digit number n. Now, if $k = 4$, we have $n' = 26$, and so $n = 26d^2$ for some positive integer d. The only possibility here is that $d = 1$, and so $n = 26$. However, we then get that $kn = 4 \cdot 26 = 104$, which is not a two-digit number.

This leaves only the possibility $k = 8$, and $n' = 3$. We must have $n = 3d^2$ for some positive integer d; since n is a two-digit number we must have $d \geq 2$. However, $kn = 8 \cdot 3d^2 = 24d^2$ is also a two-digit number, and so $d \leq 2$ as well. Thus in fact $d = 2$, and indeed we get $AB = n = 12$, $CD = 8n = 96$, and $ABCD = 1296 = 36^2$. We have $A + B + C + D = 1 + 2 + 9 + 6 = 18$.

Problem 20. Leda squares a positive integer and observes that its first (that is, leftmost) X digits are all the same (and nonzero). What is the largest possible value of X?

(A) 2 (B) 3 (C) 4 (D) 5 (E)$^\heartsuit$ There is no largest

Solution. In general, the interval starting with n 1s and ending with m arbitrary digits can be written as

$$\left[\frac{10^n - 1}{9} 10^m, \left(\frac{10^n - 1}{9} + 1\right) 10^m\right).$$

We can guarantee a square in this interval if the difference in the square roots of the upper and lower endpoints is bigger than 1 (as that would mean there is some integer between their roots). However for a fixed n, the difference in the square roots is proportional to $10^{m/2}$ which can be made arbitrarily large by choosing m large. Hence there is no limit to the number of leading identical digits of a square.

Actually, it's also possible to give an explicit construction of a number with n

leading 1s. Note

$$\begin{aligned}\underbrace{33\ldots3}_{n-1}4^2 &= \left(\frac{10^n+2}{3}\right)^2 \\ &= \frac{1}{9}\left(10^{2n}+4\cdot10^n+4\right) \\ &= \frac{10^{2n}-1}{9}+\frac{4\left(10^n-1\right)}{9}+1 \\ &= \underbrace{11\ldots1}_{n}\underbrace{55\ldots5}_{n-1}6\end{aligned}$$

Problem 21. Edna squares a positive integer and observes that its last (that is, rightmost) X digits are all the same (and nonzero). What is the largest possible value of X?

(A) 2 (B)$^\heartsuit$ 3 (C) 4 (D) 5 (E) There is no largest

Solution. Let's think about what the possible digit D is, if we want X large. By computing the squares of the numbers 1, 2, ..., 9, one sees that to have $X \geq 1$ copies of the digit D at the end of a square, one needs D to be one of 1, 4, 5, 6, or 9.

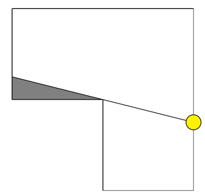
Suppose $X \geq 2$. Then our square has the form $100M + DD$ for some integer M. Looking mod 4 gives that $n \equiv DD \pmod 4$. So DD better be a square mod 4. This rules out $D = 1$, 5, 6, and 9. So the only possibility is $D = 4$.

It **is possible** for a square to end in 444: $38^2 = 1444$. So $X \geq 3$ is possible.

However, $X \geq 4$ is impossible. In order to have $X \geq 4$, we would need $D = 4$, and our square n^2 (say) would end in 4444. That is, $n^2 = 10000M + 4444$ for some M. Clearly, n is even, say $n = 2m$. But then $m^2 = 2500M + 1111$, yielding a square ending in 11. But we ruled this out above (see the $X \geq 2$ discussion).

This completes the proof, but there's still the question of how the 38 was discovered. Here's one "story" for this. It's easy to find a square that ends in 44, namely 12^2. It's then clear that the square of $100M+12$ will always end in 44. Let's look at these numbers a little more closely. FOILing out, $(100M+12)^2 = 10000M^2+2400M+144$. Taking this mod 1000, the last three digits are the same as those of $400M+144$. Subtracting the 44 and dividing by 100, we find that the digit in the hundreds place of our square is congruent to $4M+1 \pmod{10}$. This is a bust as far as finding a square that ends in 444, since we can't make $4M+1 \equiv 4 \pmod{10}$. But we can be slightly more clever and observe that the same idea works for $50M + 12$. Doing the algebra, we find the hundreds digit of our square is now congruent mod 10 to $5M^2 + 2M + 1$. And we **can** choose M to make that expression congruent to 4 mod 10; any $M \equiv -1 \pmod{10}$ will work. So for example, $M = 9$ is ok, and $50 * 9 + 12 = 462$ has a square ending in 444 (its square turns out to be 213444). OK, what about 38? Well, one can be just a little cleverer still and notice the square of $50M - 12$ also always ends in 44. One can carry out the calculations as above, or one can just try $M = 1$ and notice it works immediately.

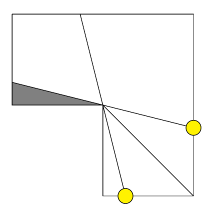
Problem 22. Consider an L shaped room made of three square parts arranged as in the picture below. If the walls do not reflect light, what is the expected proportion of the room which is lit when a bulb is placed randomly inside the room. In the example illustrated below, the bulb illuminates $\frac{23}{24}$ of the room.



(A) $\frac{5}{6}$ (B)$^{\heartsuit}$ $\frac{8}{9}$ (C) $\frac{11}{12}$ (D) $\frac{14}{15}$ (E) $\frac{17}{18}$

Solution. Let us decompose the room into the three squares that make it. The probability to have the bulb in any of those squares is $\frac{1}{3}$.
If the bulb is placed in the upper right square, it lights the whole room. Take a bulb (b) in the lower right square and consider its reflection (b') over the diagonal going from the NW corner to the SE corner (see image below). Both these bulbs completely light the lower right square and the upper right square.

By symmetry, they light on average half of the upper left square. I.e. on average (b) and (b') light $\frac{1}{3} + \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{5}{6}$ of the room.
The same argument holds if the bulb is in the upper left square.
All in all, the expected proportion of the room that is lit is $\frac{1}{3} + 2 \cdot \frac{1}{3} \cdot \frac{5}{6} = \frac{8}{9}$.



Problem 23. Anna has defined a new operation, which she calls $\star$, on pairs of real numbers: if $P = (x, y)$ and $Q = (x', y')$ then she lets $P \star Q = (x'', y'')$ as

$$\begin{cases} x'' &= xx' - yy' \\ y'' &= xy' + x'y \end{cases}.$$

Lisa picks R to be the pair $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and tells Anna, "I know what the value of

$$\underbrace{R \star \cdots \star R}_{\text{2022 copies of R}}$$

is!" Can you find the result of Lisa's computation?

(A)$^\heartsuit$ $(-1, 0)$ (B) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (C) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ (D) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (E) $(1, 0)$

Solution. Notice that if P and Q lie on the unit circle, the operation defining $\star$ is the law of addition for cosines and sines. I.e. if $P = (\cos\alpha, \sin\alpha)$ and $Q = (\sin\beta, \cos\beta)$ then $P \star Q = (\cos(\alpha+\beta), \sin(\alpha+\beta))$.
Notice that Lisa's R is nothing but $(\cos\frac{\pi}{6}, \sin\frac{\pi}{6})$. Since $2022 = 6 \cdot 337$, the result of Lisa's computation is $(\cos\pi, \sin\pi)$ or $(-1, 0)$.

Problem 24. How many positive integers n with $1 \leq n \leq 2022$ are there that **cannot** be written in the form $n = k + \lfloor \log_2 k \rfloor$ for some k? Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(A) 9 (B)$^\heartsuit$ 10 (C) 11 (D) 12 (E) 13

Solution. First approach: Define the function f by $f(k) = k + \lfloor \log_2 k \rfloor$. We start by listing a few values of $f(k)$:

$$\begin{aligned} f(1) &= 1 \\ f(2) &= 2 + 1 = 3 \\ f(3) &= 3 + 1 = 4 \\ f(4) &= 4 + 2 = 6 \\ &\vdots \end{aligned}$$

Observe that all the jumps are happening at the powers of 2. We have $f(2^m) = 2^m + m$, $f(2^m - 1) = (2^m - 1) + (m-1) = 2^m + m - 2$. That is, we are losing numbers of the form $2^m + m - 1$. To show that these are the only ones we lose, observe that any positive integer n can be written in the form $2^m + d$ for some $1 \leq d \leq 2^m$. If $m \leq d \leq 2^m$, we have $2^m + d = 2^m + (d-m) + m$; we then have $f(2^m + d - m) = 2^m + d = n$ as desired. On the other hand, if $0 \leq d < m-1$, we have $2^m + d = 2^m + d - (m-1) + (m-1)$; then we have, since $d - (m-1) < 0$, $f(2^m + d - (m-1)) = 2^m + d - (m-1) + (m-1) = 2^m + d$. However, there is no k for which $f(k) = 2^m + m - 1$. This is because if $k < 2^m$, we have $f(k) \leq (2^m - 1) + (m-1) = 2^m + m - 2$; on the other hand, if $k \geq 2^m$, we must have $f(k) \geq 2^m + m$.

The only thing left to do is count all integers $n = 2^m + m - 1$ with $1 \leq n \leq 2022$. We have at $m = 10$, $2^{10} + 9 = 1033 < 2022$, but at $m = 11$ we have $2^{11} + 10 = 2058 > 2022$. Thus there are 10 such integers.

Second approach: Define the function f as in Solution 1. Observe that f is a strictly increasing function, and so $f(1) < f(2) < \ldots$ Now, notice that $f(2048) = f(2^{11}) = 2^{11} + 11 = 2059$ is a little higher than 2022, but not too much. Moving a little lower, we have $f(2047) = 2047 + 10 = 2057$, $f(2046) = 2046 + 10 = 2056$, and so on and so forth; it seems reasonable to surmise that $f(k)$ goes down by 1 most of the time when k does, and thus starting from $f(2047) = 2057$, going down by $2057 - 2022 = 35$ gives us $f(2047 - 35) = f(2012) = 2012 + 10 = 2022$. Thus we have $1 = f(1) < f(2) < \cdots < f(2012) = 2022$ are 2012 integers n with $1 \leq n \leq 2022$

of the form $n = f(k)$. Since f is increasing, for any $k > 2012$ we have $f(k) > 2022$; these are thus in fact the only such integers. That is, there are 2012 integers n with $1 \leq n \leq 2022$ such that $n = f(k)$ for some k, and so there are $2022 - 2012 = 10$ that are not.

Problem 25. A magic square is a grid with numbers such that the sum of all entries in any given row, column or diagonal is a fixed constant.

Consider a magic square, all whose entries are strictly positive integers (but not necessarily distinct), and of which only one of the entries is known:

X		
	9	

What is the minimal possible value of the entry in the upper left corner?

(A) 1 (B) 2 (C) 3 (D) 4 (E)$^\heartsuit$ 5

Solution.

Let us label the squares as below :

X	A	B
	C	D
	9	E

The following is a somewhat magical identity: Note

$$(A + C + 9) + (B + D + E) = (X + A + B) + (X + C + E)$$

reduces to $9 + D = 2X$, so X is the average of 9 and D. Since D is at least 1, we have $X \geq 5$. To see that this is actually attainable, here is a valid magic square.

5	1	9
9	5	1
1	9	5

Note: In practice, there are a lot of ways to equation bash to find that $9+D = 2X$, the above solution method is just the shortest such way the authors were able to find.

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