Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 26, 2019

WITH SOLUTIONS

Problem 1. How many of the following equations have solutions in positive integers?

i) $28x + 30y + 31z = 365$
ii) $29x + 30y + 31z = 366$
iii) $10x + 26y = 2019$
iv) $3x + 4y = 5z$

(A) 0 (B) 1 (C) 2 (D)$^\heartsuit$ 3 (E) 4

Solution. For equations (i) and (ii), think about days of the months and years. For (iii) notice that the left side is even but the right is odd. (iv) is reminiscent of a famous Pythagorean triple.

Problem 2. Suppose that a, b, and c are three distinct real numbers. For each nonempty subset of $\{a, b, c\}$, we can compute its average. For example,

$$\text{average}\{a\} = a,$$

$$\text{average}\{a, b\} = \frac{a+b}{2}.$$

If the averages, listed in increasing order, of the 7 nonempty subsets of $\{a, b, c\}$ are

$$2, 6, 10, 12, 13, 17, 24,$$

then what is $\dfrac{a+b+c}{3}$?

(A) 6 (B) 10 (C)$^\heartsuit$ 12 (D) 13 (E) 17

Solution. Assuming $a < b < c$, we must have $a = 2$ and $c = 24$, and so $\frac{a+c}{2} = 13$. Since $b > a$, we see that $\frac{b+c}{2} > \frac{a+c}{2}$, so $\frac{b+c}{2} = 17$, and $b = 10$. Thus,

$$\frac{a+b+c}{3} = \frac{2+10+24}{3} = 12.$$

Alternatively, notice that if you sum the seven given numbers, that sum will include a once (from $\{a\}$), $a/2$ twice (from $\{a,b\}$ and $\{a,c\}$) and $a/3$ once; b and c act similarly. Therefore,

$$a + 2\cdot\frac{a}{2} + \frac{a}{3} + b + 2\cdot\frac{b}{2} + \frac{b}{3} + c + 2\cdot\frac{c}{2} + \frac{c}{3} = 2+6+10+12+13+17+24.$$

Collecting like terms, $\frac{7}{3}(a+b+c) = 84$, and so $\frac{a+b+c}{3} = 12$.

Problem 3. If $5^x \cdot 7^y = 4$, then y, as a function of x, is

(A)$^\heartsuit$ linear (B) polynomial of degree ≥ 2 (C) exponential
(D) logarithmic (E) y is not a function of x

Solution. We solve for y by taking the natural logarithm of both sides:

$$\begin{aligned} 5^x \cdot 7^y = 4 &\Longleftrightarrow \ln(5^x \cdot 7^y) = \ln 4 \\ &\Longleftrightarrow \ln(5^x) + \ln(7^y) = \ln 4 \\ &\Longleftrightarrow x \ln 5 + y \ln 7 = \ln 4. \end{aligned}$$

Thus, $y = \frac{1}{\ln 7}(\ln 4 - x\ln 5)$, which is linear in x with slope $-\ln(5)/\ln(7)$.

Problem 4. Tank A is a vertical cylinder with radius 2 ft; tank B is a vertical cylinder with radius 3 ft. Water is drained from tank A into tank B. If the depth of the water in tank A is decreasing at a constant rate of 1 ft/hr, at what rate is the depth of the water in tank B increasing?

(A) 1 ft/hr (B) $\dfrac{2}{3}$ ft/hr (C) $\dfrac{3}{2}$ ft/hr (D)$^\heartsuit$ $\dfrac{4}{9}$ ft/hr (E) $\dfrac{9}{4}$ ft/hr

Solution. If the height in tank A is decreasing at a constant rate of 1 ft/hr, then the volume lost per hour is $\pi \cdot 2^2 \cdot 1 = 4\pi$ ft^3/hr. So the volume gained in tank B is 4π ft^3/hr, which is enough to fill a cylinder of radius 3 and height 4/9 every hour.

Problem 5. If you cut a finite interval randomly into two pieces, what is the probability that one of them is at least three times as long as the other?

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{3}$ (C)$^\heartsuit$ $\dfrac{1}{2}$ (D) $\dfrac{2}{3}$ (E) $\dfrac{3}{4}$

Solution. Imagine the interval as $[0,1]$ and cutting at point x. Then one piece is at least three times as long as the other if (and only if) $0 \leq x \leq \frac{1}{4}$ or $\frac{3}{4} \leq x \leq 1$. So the probability is $\frac{1}{2}$.

Problem 6. What is the smallest n such that a group of n people has probability $\geq \frac{1}{2}$ of having two people born in the same month, assuming that each of the 12 possible birth months is equally likely?

(A) 3 (B) 4 (C)$^\heartsuit$ 5 (D) 6 (E) 7

Solution. We begin by computing the probability q that a group of n people all have *distinct* birth months. We are looking for the smallest n with $q \leq \frac{1}{2}$. For $n \leq 12$ people, the probability of distinct birth months is

$$q = q(n) = \frac{12}{12} \cdot \frac{11}{12} \cdot \frac{10}{12} \cdots \frac{12-(n-1)}{12}.$$

Notice that if $n = 5$, then $q = \frac{12}{12} \cdot \frac{11}{12} \cdot \frac{10}{12} \cdot \frac{9}{12} \cdot \frac{8}{12}$, and $\frac{9}{12} \cdot \frac{8}{12} = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$, so that $q(5) < \frac{1}{2}$. Then check that $q(4) = \frac{55}{96} > \frac{1}{2}$.

Why begin by computing the probability that the desired property does *not* occur? Notice that "n people have distinct birth months" is a conjunction ('and') of independent events, while "2 people have the same birth month" is a significantly more complicated disjunction ('or' statement).

Problem 7. If $f(x) = e^{-x}$, what is the range of $f(f(f(x)))$?

(A) $(1, e)$ (B)$^\heartsuit$ $(1/e, 1)$ (C) $(0, 1/e)$ (D) $(0, 1)$ (E) $(0, e)$

Solution. If we denote by $g(A)$ the set of values arising as $g(x)$, for $x \in A$, then $g(\mathbb{R})$ is the range of g. So we want $f(f(f(\mathbb{R})))$. We have

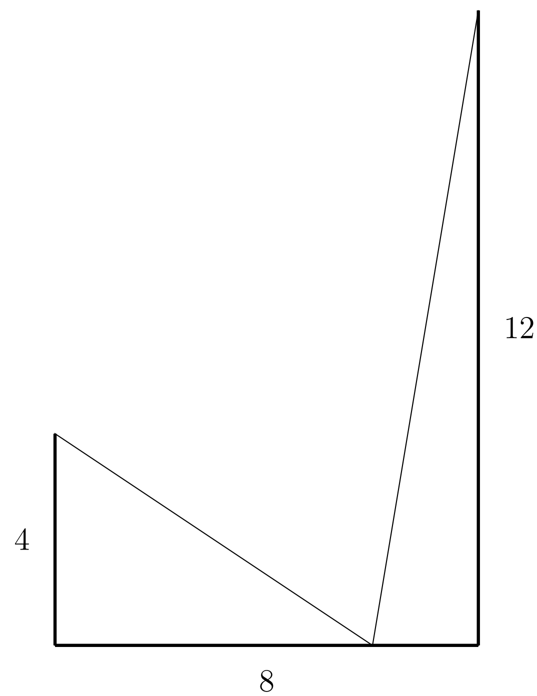
$$f(\mathbb{R}) = \{e^{-x} : x \in \mathbb{R}\} = (0, \infty),$$

$$f(f(\mathbb{R})) = \{e^{-x} : x \in (0, \infty)\} = (0, 1),$$

and

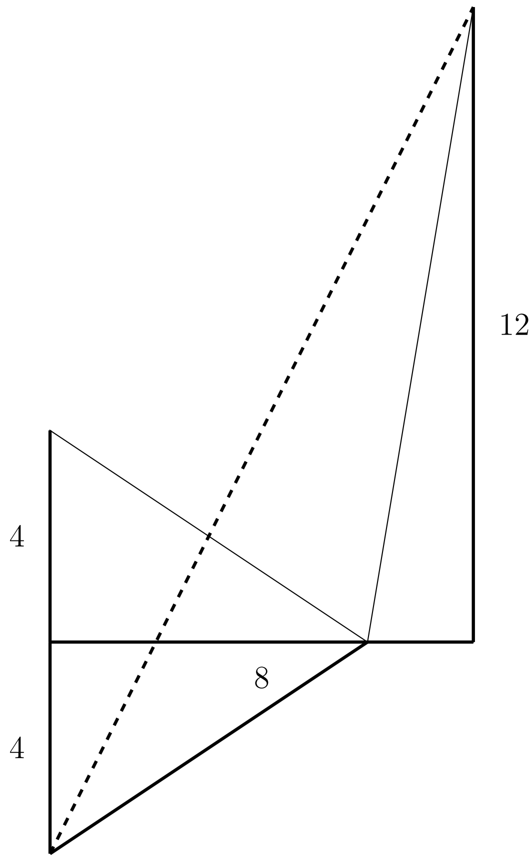
$$f(f(f(\mathbb{R}))) = \{e^{-x} : x \in (0, 1)\} = (1/e, 1).$$

Problem 8. You are connecting a wire between the top of a height 4 ft pole and the top of a height 12 ft pole. The wire also has to be connected to the ground between the poles. If there is 8ft between the poles, what is the minimum length of the wire?



(A)$^\heartsuit$ $8\sqrt{5}$ (B) 18 (C) $13\sqrt{2}$ (D) 20 (E) 21

Solution. If we reflect the first pole through the ground, the wire corresponds to a path between the tip of the underground pole and the tip of the above ground pole. Since the shortest path between two points is a straight line, the shortest length of wire possible is $\sqrt{8^2+16^2} = 8\sqrt{1^2+2^2} = 8\sqrt{5}$.



Problem 9. Suppose each positive integer is assigned exactly one of the colors red, black or silver, in such a way that for every $z \geq 3$ there are distinct positive integers x and y so that $x + y = z$, and x, y, and z are all different colors; i.e. $x + y = z$ has a "rainbow solution." If 2019 is red, what color(s) can 1 be?

(A) red (B) black (C)$^\heartsuit$ black or silver (D) red, black, or silver
(E) there is no such coloring

Solution. Since $1 + 2 = 3$ is the only way to write 3 as the sum of distinct positive integers, each of those integers must be assigned a different color. Let's say 1 has color a, 2 has color b, and 3 has color c.

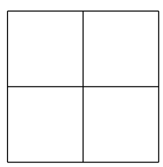
We shall show that the only possible coloring is: 1 has color a, every odd number $\neq 1$ has color c, and every even number has color b. Let us call this coloring $(*)$.

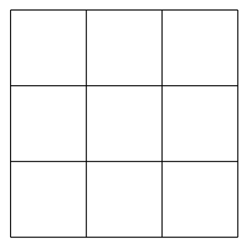
Notice that we have shown 1, 2, and 3 must satisfy $(*)$. Now suppose $n > 3$ is even and k satisfies $(*)$ for every $1 \leq k < n$. Then n can be written as either the sum of two even numbers or the sum of two odd numbers. In the former case, all even numbers smaller than n have the same color, so they can't form a rainbow solution with n. Similarly, in the latter case all odd numbers except 1 have the same color, so the only possibility for a rainbow solution is $1 + (n - 1) = n$, which forces n to have color b as 1 has color a and $(n - 1)$ has color c. Thus if n is even, then n satisfies $(*)$.

Now suppose $n > 3$ is odd, and k satisfies $(*)$ for every $1 \le k < n$. Then n can only be written as the sum of an even number and an odd number. This seems to give two possibilities for the color of n as $3 + (n-3) = n$ allows the color a (as 3 has color c and $n-3$ has color b), and $1 + (n-1) = n$ allows the color c (as 1 has color a and $n-1$ has color b). However if n has color a, then there is no possible color for $n+1$: Since $n+1$ is even, we have already seen the rainbow solution must come from $1 + n = n + 1$, but now 1 and n both have color a, leading to an impossibility. Therefore if n is odd, it must also satisfy $(*)$.

Noting that $(*)$ is indeed a valid coloring from the equation $1 + (z-1) = z$ for $z \ge 3$, we conclude that if 2019 is red, then 1 cannot be colored red, but it could be colored either silver or black.

Note. For $n > 1$, a geometric figure is called an *n-reptile* if it can be decomposed into n figures which are congruent to each other and similar to the original figure. For example, a square is a 4-reptile, and also a 9-reptile.





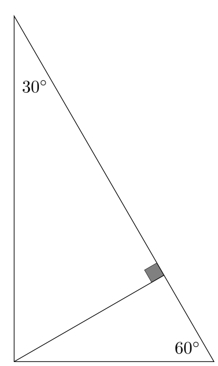
You also saw in the ciphering round that a $1 \times \sqrt{2}$ rectangle is a 2-reptile.

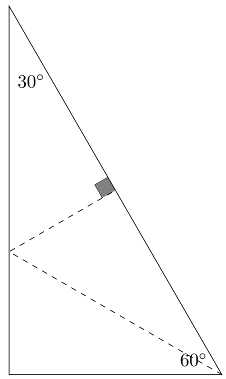
Problem 10. What is the smallest $n > 1$ for which a 30°-60°-90° triangle is an n-reptile?

(A) 2 (B)$^\heartsuit$ 3 (C) 4 (D) 5 (E) 6

Solution. *Every* triangle is a 4-reptile: Connect the midpoints of the sides. So $n \le 4$.

A 30°-60°-90° triangle is not a 2-reptile; notice that in the decomposition shown here, the smaller triangles are similar to the larger, but not congruent to each other.



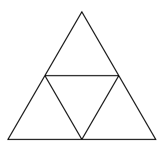


Since 30° is half of 60°, a 30°-60°-90° triangle can be decomposed into three congruent triangles by bisecting the 60°-degree angle. Thus, $n = 3$.

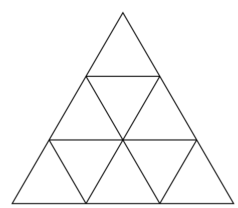
Problem 11. For how many n, $1 < n < 50$, is an equilateral triangle an n-reptile?

(A) 2 (B) 3 (C) 4 (D) 5 (E)$^{\heartsuit}$ 6

Solution. We have already seen that any equilateral triangle (indeed, any triangle) is a 4-reptile, by connecting the midpoint of the sides. Similarly, by connecting the trisectors of the sides, with lines parallel to the sides, we see that an equilateral triangle is a 9-reptile. Continuing in the same way, we find that an equilateral triangle is an n^2-reptile for any $n > 1$. In particular, it is an n-reptile for at least 6 values of n, $1 < n < 50$: $4, 9, 16, 25, 36, 49$.



$n = 3$



$n = 9$

Here is an argument that n must be a square. Let x be the side length of the equilateral triangle, and let s be the side length of a reptile piece. Then x/s must be an integer as this is the number of pieces with an edge on a single side of the triangle. However, area considerations tell us $x^2\sqrt{3}/4 = n(\sqrt{3}/4)s^2$, so $n = (x/s)^2$, and n is a square.

Problem 12. What is the smallest positive integer n such that $p(n) = n^2 + 20n + 19$ is divisible by 2019?

(A) 654 (B) 672 (C) 1327 (D)$^{\heartsuit}$ 2000 (E) 2019

Solution. We need $2019 = 3 \cdot 673$ to divide $n^2 + 20n + 19 = (n + 1)(n + 19)$. This certainly happens when $n = 2000$, since then $n+19 = 2019$. To see this is the smallest n, we argue as follows. Since $3 \mid (n + 1)(n + 19)$, and 3 is prime, 3 divides at least one of $n + 1$ and $n + 19$. But $n + 19$ and $n + 1$ differ by $3 \cdot 6$, and so in fact 3 divides both of $n+1$ and $n+19$. Since 673 is prime, it also divides at least one of $n+1$ and $n+19$. Putting these facts together, the number $3 \cdot 673 = 2019$ divides either $n+1$ or $n+19$. But when $n < 2000$, both numbers are too small to be multiples of 2019.

Problem 13. For how many positive integers n is the decimal expansion of $\frac{1}{n}$ purely periodic with period 3 (and no smaller period!) ?

(A) 4 (B)$^\heartsuit$ 5 (C) 6 (D) 7 (E) 8

Solution. Say $\frac{1}{n} = 0.\overline{abc}$, with $a, b, c \in \{0, 1, \ldots, 9\}$. Then

$$1000 \cdot 0.\overline{abc} - 0.\overline{abc} = abc,$$

so that $0.\overline{abc} = abc/999$. Thus, $999/abc = n$, and so abc is a factor of 999. Since 999 factors into primes as $3^3 \cdot 37$, the divisors of 999 are $3^a \cdot 37^b$ where $a \in \{0, 1, 2, 3\}$ and $b \in \{0, 1\}$. There are $4 \cdot 2 = 8$ such divisors, and so 8 numbers n for which $1/n$ has period 3. To ensure that 3 is the *smallest* period, we have to remove from our list of divisors numbers n that are factors of $99 = 3^2 \cdot 11$ (corresponding to period 2) or factors of $9 = 3^2$ (period 1). This rules out $1, 3$ and 9, leaving 5 possibilities.

Problem 14. If $a = 1 + z$ and $b = 1 - \frac{1}{z}$ and $a + b = 7$, what is $a^3 + b^3$?

(A)$^\heartsuit$ 238 (B) 288 (C) 321 (D) 325 (E) 343

Solution. Notice that $ab = z - \frac{1}{z} = a + b - 2 = 5$. Therefore

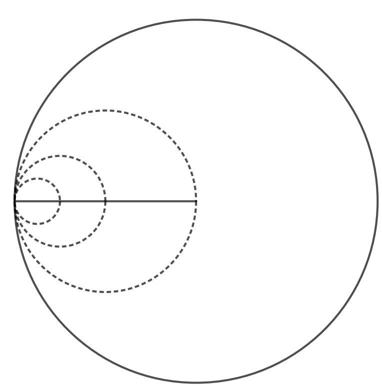
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (a + b)((a + b)^2 - 3ab) = 7(49 - 15) = 238.$$

Note. The next two problems are about the area of a union of disks. Recall that a *disk* consists of a circle together with the points inside the circle.

Problem 15. Let L be the line segment in $\mathbb{R}^2$ from $(0, 0)$ to $(2, 0)$. At each point $(x, 0)$ of L draw a disk of radius x centered at $(x, 0)$. What is the area of the union of these disks?

(A) $2 + \dfrac{\pi}{2}$ (B) $2 + \pi$ (C) $2 + 2\pi$ (D) 2π (E)$^\heartsuit$ 4π

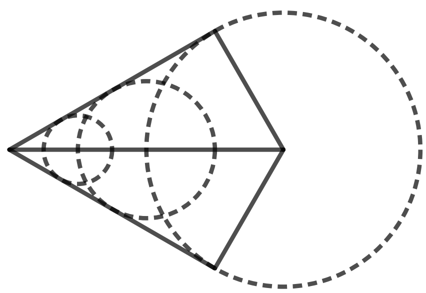
Solution. These disks are nested as shown, so their union is the largest disk, which has radius 2 and area 4π.

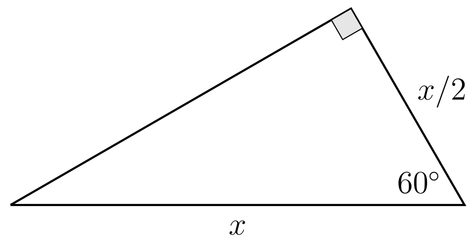


Problem 16. Again let L be the line segment in $\mathbb{R}^2$ from $(0,0)$ to $(2,0)$. At each point $(x,0)$ of L, draw a disk of radius $\frac{x}{2}$ centered at $(x,0)$. What is the area of the union of these disks?

(A) $\dfrac{\sqrt{3}}{2}+\pi$ (B) $\sqrt{3}+\pi$ (C) $\dfrac{\sqrt{3}}{2}+\dfrac{2}{3}\pi$ (D)$^{\heartsuit}$ $\sqrt{3}+\dfrac{2}{3}\pi$ (E) 2π

Solution. Each of the right triangles drawn has hypotenuse x and leg $x/2$, and so is a 30°-60°-90° right triangle. So this region consists of two right triangles, each of area $\frac{1}{2}(1)\sqrt{3}$, together with $\frac{2}{3}$ of a disk of radius 1, for a total area of $\sqrt{3}+\frac{2}{3}\pi$.





Problem 17. The terms in a sequence of positive integers add up to 2019. If P is the largest possible product of these terms, what is the number of different primes dividing P?

(A)$^{\heartsuit}$ 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution. We solve the analogous problem for any integer $n>1$, not simply 2019.

First, we show that if S is a sequence summing to n with maximal product, then every term of S is $2,3$, or 4. If 1 is a term of S, choose any other term m. (We use here that $n>1$.) Replacing 1 and m with the single term $m+1$ does not change the sum but increases the product, contradicting the choice of S. So 1 cannot be a term. Similarly, if S has any terms $m>4$, replacing m with the two terms 2 and $m-2$ preserves the sum but increases the product.

Next, we can replace every 4 in S with two 2s, preserving both the sum and product. Hence, there is a sequence S' summing to n with largest possible product and consisting of only 2s and 3s.

Continuing, notice that there can be at most two 2s in S': If there are three 2s, replacing these with two 3s preserves the sum but results in a larger product.

Let A and B be the number of 2s and 3s in S', so that $0\le A\le 2$. Since $n=2A+3B$, we see that A is the unique integer in $\{0,1,2\}$ with $n-2A$ a multiple of 3, and $P=2^A3^B$.

When $n=2019$, we have $A=0$ and $B=673$. Thus, $P=3^{673}$, which has 3 as its unique prime divisor.

Problem 18. Recall that a polygon is said to be *convex* if all of its interior angles measure strictly less than $180°$. A *lattice point* in $\mathbb{R}^2$ is a point with coordinates (x, y), with x and y both integers. Now consider the following statement:

> Every convex n-gon in $\mathbb{R}^2$ having all its vertices at lattice points contains a lattice point in its interior.

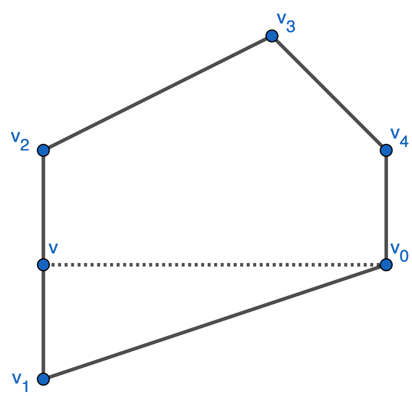
The least positive integer $n \geq 3$ for which this statement holds is

(A) 4 (B)$^\heartsuit$ 5 (C) 6 (D) 7 (E) there is no such n

Solution. The statement is false for $n = 3$; consider the triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. It is also false for $n = 4$; consider the square with vertices $(0,0), (1,0), (1,1)$, and $(0,1)$. But it is true when $n \geq 5$.

To prove this, we proceed in two steps. First, we show that it is enough to prove the result in the special the case where each segment between consecutive vertices is free of lattice points.

Let P be an n-gon satisfying our assumptions, but with a lattice point somewhere between adjacent vertices. Going around clockwise, label the vertices as v_0, v_1, v_2, $\ldots$, v_{n-1}, so that P is formed by connecting v_0 to v_1, v_1 to v_2, $\ldots$, and v_{n-1} to v_0. We can choose the starting vertex for our numbering so that there is a lattice point v between v_1 and v_2.



Now let P' be the n-gon with vertices $v_0, v, v_2, \ldots, v_{n-1}$. Then P' is an n-gon satisfying our initial hypotheses. Moreover, if P' has a lattice point between any two of its adjacent vertices, we can repeat the process. Every iteration results in a strictly smaller polygon, and so decreases the enclosed area. But (e.g., by Pick's theorem), this area is always half an integer, so the process terminates in a polygon with no lattice points between consecutive vertices.

We now finish the proof. Let P be a polygon satisfying our assumptions, with the extra condition that there is no lattice point between consecutive vertices. Assign each lattice point a 'parity', namely the ordered pair consisting of the parity of its components. There are only 4 possible parities of lattice points: (even, even), (even, odd), (odd, even), and (odd, odd). So among the $n \geq 5$ vertices of P, there are two, say $v = (x, y)$ and $v' = (x', y')$, sharing the same parity. Then the midpoint of v and v' is a lattice point. Thus, v and v' cannot be consecutive! Now convexity implies that the segment connecting v and v' lies entirely inside P, and so this midpoint is an interior lattice point.

Problem 19. For how many positive integers n is $(3.5)^n + (12.5)^n$ an integer?

(A) 1 (B) 2 (C)$^\heartsuit$ 3 (D) 4 (E) infinitely many

Solution. Asking for $(3.5)^n + (12.5)^n$ to be integer is equivalent to asking that $2^n \mid 7^n + 25^n$. If n is even, then 7^n and 25^n are both 1 mod 4, so their sum is divisible by 2 but not by 4. Hence, there are no even positive integers n for which $2^n \mid 7^n + 25^n$. Suppose that n is odd. Then

$$\begin{aligned} 7^n + 25^n &= (7 + 25) \cdot (7^{n-1} - 25 \cdot 7^{n-2} + (25)^2 \cdot 7^{n-3} - \cdots + 25^{n-1}) \\ &= 2^5 \cdot (7^{n-1} - 25 \cdot 7^{n-2} + (25)^2 \cdot 7^{n-3} - \cdots + 25^{n-1}). \end{aligned}$$

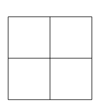
The parenthesized right-hand factor is a sum of n odd numbers, and so is odd. Thus, 2^5 is always the highest power of 2 dividing $7^n + 25^n$, when n is odd. So the odd n with $2^n \mid 7^n + 25^n$ are $n = 1, 3$, and 5.

Problem 20. Suppose you want to cover an 8×8 checkerboard with 21 L-shaped tiles and a single 1×1 square tile.

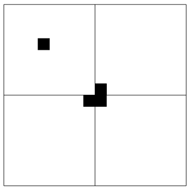
How many different locations are there on the board where the square tile could be located in a successful tiling? You are allowed to rotate the L-shaped tile.

(A) 0 (i.e., no such tiling is possible) (B) 4 (C) 8 (D) 22 (E)$^\heartsuit$ 64

Solution. Here is a proof by induction that a $2^n \times 2^n$ checkerboard can be tiled by $\frac{4^n-1}{3}$ L-shaped tiles and 1 square tile, regardless of where the square is placed.



BASE CASE: $n = 1$.
One L-tile and one square can tile the board regardless of where it is placed.
INDUCTION STEP: Given a $2^{n+1} \times 2^{n+1}$ board, place the square tile anywhere and partition the board into four $2^n \times 2^n$ boards.



One of these contains the square tile. Now place an L-tile on the 3 corners where the other 3 smaller boards meet. All that remains is to tile these $2^n \times 2^n$ boards, each with one square already covered. This can be done by the induction hypothesis.

Notice that this proof describes an algorithm for tiling a board. This is in fact the only way to tile 2×2 and 4×4 boards.

Challenge. Is there any other way to tile an 8×8 (or larger) board?

Problem 21. Suppose you want to cover an 8×8 checkerboard with 21 straight tiles of length 3 and a single 1×1 square tile.

How many different locations are there on the board where the square tile could be located in a successful tiling? You are allowed to rotate the straight tile.

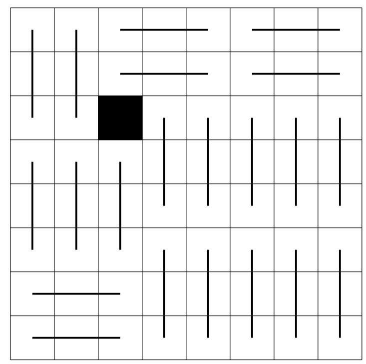
(A) 0 (i.e., no such tiling is possible) (B)$^\heartsuit$ 4 (C) 8 (D) 22 (E) 64

Solution. Number the squares of the board as shown here. Then:

- The sum of all of the numbers is 128.
- Any 3-tile placed on the board will cover $1, 2, 3$ (in some order).
- The 21 3-tiles will cover a total of $21 \cdot 6 = 126$, so the square tile must lie on a 2. There are 22 of these on the board as it is numbered.
- If we rotate the numbering by $90°$, we get a different numbering and the square must still lie on a 2.
- There are only 4 squares that are on a 2 in both numberings.

1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3

Finally, each of these 4 squares can be the position of the square tile, as shown by the following diagram and its $90°$ rotations:



Problem 22. For each integer $n > 1$, let $f(n)$ denote the number of *factorizations* of n, meaning the number of ways of writing n as an ordered product of integers larger than 1. For example, 12 has 8 factorizations:

$$12, \quad 2 \cdot 6, \quad 6 \cdot 2, \quad 3 \cdot 4, \quad 4 \cdot 3, \quad 2 \cdot 2 \cdot 3, \quad 2 \cdot 3 \cdot 2, \quad 3 \cdot 2 \cdot 2.$$

Let $f_{\text{odd}}(n)$ count the number of factorizations with an odd number of parts and $f_{\text{even}}(n)$ count the number of factorizations with an even number of parts. What is the largest value of

$$|f_{\text{odd}}(n) - f_{\text{even}}(n)|$$

for $1 < n \leq 2019$?

(A) 0 (B)$^{\heartsuit}$ 1 (C) 3 (D) 39 (E) 40

Solution. We will show that for every $n > 1$, the difference $f_{\text{even}}(n) - f_{\text{odd}}(n) = \mu(n)$, where $\mu(n)$ is the so-called Möbius function, defined by

$$\mu(n) = \begin{cases} 0 & \text{if } p^2 \mid n \text{ for some prime } p, \\ (-1)^k & \text{if } n = p_1 \cdots p_k \text{ for distinct primes } p_i. \end{cases}$$

Thus, $f_{\text{even}}(n) - f_{\text{odd}}(n)$ assumes only the values $-1, 0, 1$.

In what follows, we think of 1 as an empty product. In keeping with this, we adopt the conventions that $\mu(1) = 1$, $f(1) = 1$, $f_{\text{even}}(1) = 1$, and $f_{\text{odd}}(1) = 0$.

To begin the proof, observe that for each $n > 1$,

$$f(n) = \sum_{d \mid n,\ d < n} f(d).$$

To see this, think of constructing a factorization of n by first factoring d, for a proper divisor d of n, and then appending a final term of n/d. This same argument implies that for all $n > 1$,

$$f_{\text{odd}}(n) = \sum_{d \mid n,\ d < n} f_{\text{even}}(d)$$

and

$$f_{\text{even}}(n) = \sum_{d|n,\ d<n} f_{\text{odd}}(d).$$

Now let $g(n) = f_{\text{even}}(n) - f_{\text{odd}}(n)$. Then $g(1) = 1$ and (from the last two displayed equations) we have for every $n > 1$ that

$$g(n) = -\sum_{d<n} g(d).$$

Rearranging,

$$\sum_{d|n} g(d) = 0$$

for every $n > 1$.

Remember that we are trying to show that $g = \mu$. An easy induction shows that g is uniquely determined by the conditions that (a) $g(1) = 1$ and (b) $\sum_{d|n} g(d) = 0$ for every $n > 1$. Since $\mu(1) = 1$, we will have the desired identity once we prove that

$$\sum_{d|n} \mu(d) = 0$$

for every $n > 1$.

Factor $n = p_1^{e_1} \cdots p_k^{e_k}$. If $d \mid n$ and d is not squarefree, then $\mu(d) = 0$, and d does not contribute to $\sum_{d|n} \mu(d)$. So it is enough to consider the contribution of d dividing $p_1 \cdots p_k$. For every ℓ, there are $\binom{k}{\ell}$ integers d dividing $p_1 \cdots p_k$ with ℓ prime factors, and each such d makes a contribution of $(-1)^\ell$. Thus,

$$\sum_{d|n} \mu(d) = \sum_{\ell \geq 0} (-1)^\ell \binom{k}{\ell} = (1-1)^k = 0^k = 0,$$

where we used the binomial theorem to evaluate the sum on ℓ.

Problem 23. Recall: If X and Y are sets, we say X is a *subset* of Y if every element of X is also an element of Y; we denote this by $X \subseteq Y$. The empty set $\emptyset$ is a subset of every set. We say X is a *proper subset* of Y if X is a subset of Y and $X \neq Y$; in this case, we write $X \subsetneq Y$.

How many triples of sets A, B, C are there that satisfy

$$\emptyset \subseteq A \subsetneq B \subsetneq C \subseteq \{1, 2, 3, 4, 5\} \quad ?$$

(A) 504 (B)$^\heartsuit$ 570 (C) 729 (D) 768 (E) 776

Solution. We first count the number of triples satisfying the relaxed condition

$$\emptyset \subseteq A \subseteq B \subseteq C \subseteq \{1, 2, 3, 4, 5\}.$$

The answer to this modified problem is 4^5, or 1024. For each of $1, 2, 3, \ldots, 5$, there are 4 possibilities. Either they belong to none of C, B, A, they belong only to C, they belong only to C and B, or they belong to all of C, B, and A.

To solve the problem as stated, we do an inclusion-exclusion. The number of triples with $A = B$ is the number of pairs of sets A, B with $\emptyset \subseteq A \subseteq C \subseteq \{1, 2, 3, 4, 5, 6\}$, which — by reasoning analogous to the above — is $3^5 = 243$. So we should subtract 243. We should also subtract 243 to account for triples with $B = C$. But now we need to add back in the count of triples with $A = B = C$, which is $2^5 = 32$. So our final answer is

$$1024 - 2 \cdot 243 + 32 = 570.$$

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Problem 24. How many ordered triples of integers (x, y, z) are there satisfying the simultaneous conditions

$$x + y + z = 3 \qquad \text{and} \qquad x^3 + y^3 + z^3 = 3 \quad ?$$

The integers x, y, z do not have to be distinct, and are allowed to be positive, negative, or zero.

(A) 1 (B)$^\heartsuit$ 4 (C) 7 (D) 10 (E) infinitely many

Solution. The key is to recognize the following factorization:

$$(x + y + z)^3 - (x^3 + y^3 + z^3) = 3(x + y)(x + z)(y + z).$$

Where does this come from? If we let $x = -y$, the left-hand side vanishes. So viewed as a polynomial in x, the left-hand side has $-y$ as a root, and so $x - (-y) = x + y$ is a factor. Symmetric reasoning accounts for the factors of $x + z$ and $y + z$ on the right-hand side. Comparing degrees shows that the left-hand side has the form $C(x + y)(x + z)(y + z)$ where C is a constant. To see that $C = 3$ is the correct constant, we can plug in values of x, y and z, for instance $x = y = z = 1$.

If $x + y + z = 3$ and $x^3 + y^3 + z^3 = 3$, the above factorization reveals that

$$8 = (x + y)(x + z)(y + z).$$

The condition that $x+y+z = 3$ amounts to requiring that the three right-hand factors sum to 6. The only possibilities for these factors are $2, 2, 2$ or some permutation of $8, -1, -1$. These in turn yield 4 possibilities for (x, y, z): $1, 1, 1$ or some permutation of $-5, 4, 4$.

Problem 25. Suppose $p(x)$ is a degree 6 polynomial which is tangent to $y = 2x + 3$ at $x = 1$, $x = 2$, and $x = 5$. What values can $p(0)$ have?

(A) $p(0) = 3$ (B) $p(0) = 100$ (C)$^\heartsuit$ $p(0)$ can be any real number except 3
(D) $p(0)$ can be any real number except 100 (E) there is no such polynomial

Solution. First let's see that there is such a polynomial: For any positive number k (a similar argument works for $k < 0$), notice that $k(x-1)^2(x-2)^2(x-5)^2 \geq 0$, with equality if and only if $x = 1, 2,$ or 5. This shows that $y = k(x-1)^2(x-2)^2(x-5)^2$ is tangent to the x-axis at $x = 1$, $x = 2$, and $x = 5$. Then $p_k(x) = k(x-1)^2(x-2)^2(x-5)^2 + 2x + 3 \geq 2x + 3$, with equality if and only if $x = 1, 2, 5$, so this $p(x)$ is tangent to the line $y = 2x + 3$ at the desired points — as long as $k \neq 0$! Since $p_k(0) = 100k + 3$, $p(0)$ can have any value $100k + 3$, as long as $k \neq 0$.

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Notes. Problem 12 was proposed by Nikon Kurnosov.

Problem 19 is a special case of Problem 30 in D.J. Newman's book *A Problem Seminar*.

Other solutions to Problem 22 are described in the paper

> R. Garfield, D.E. Knuth, and H.S. Wilf, *A bijection for ordered factorizations*. J. Combin. Theory Ser. A 54 (1990), 317–318.

The solution to Problem 24 is due to Andrzej Schinzel and is described in Wacław Sierpinski's book *250 Problems in Elementary Number Theory*. Until last month, the only known integer triples x, y, z with $x^3 + y^3 + z^3 = 3$ were those appearing in the problem solution. But in mid-September, Andrew Booker (U. Bristol) and Drew Sutherland (MIT) announced their discovery that

$$569936821221962380720^3 + (-569936821113563493509)^3 + (-472715493453327032)^3 = 3.$$