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Team Round, 3 problems / 1 hour / 210 points

October 21, 2023

WITH SOLUTIONS

Problem 1 (Party)**.** Mo has invited 2023 guests for his retirement party. His way of the sharing the cake is quite eccentric: the first guest gets $\frac{1}{2023}$rd of the cake, the second guest gets $\frac{2}{2023}$rd of what is left, the third guest gets $\frac{3}{2023}$rd of what is left, ..., and the last guest gets $\frac{2023}{2023}$rd -that is everything- of what is left.
Which guest receives the largest piece?

Answer. 45

Solution. Let p_n be the size of the piece of the n-th guest. Instead of looking at the problem globally, we will compare successive terms to see when we reach an apex. We have $p_n = \frac{n}{2023}\left(1 - \sum_{i=1}^{n-1} p_i\right)$. To make our life easy, and to be able to compare the sums appearing on the right hand side, let us compute

$$2023\left(\frac{p_{n+1}}{n+1} - \frac{p_n}{n}\right) = \left(1 - \sum_{i=1}^{n} p_i\right) - \left(1 - \sum_{i=1}^{n-1} p_i\right) = -p_n.$$

We can now isolate $\frac{p_{n+1}}{p_n}$:

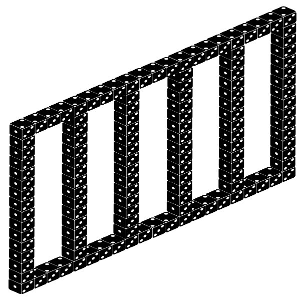
$$\frac{p_{n+1}}{p_n} = (n+1)\left(\frac{1}{n} - \frac{1}{2023}\right).$$

The sequence decreases when $\frac{p_{n+1}}{p_n} < 1$. This constraint is a quadratic inequality :

$$n^2 + n - 2023 > 0$$

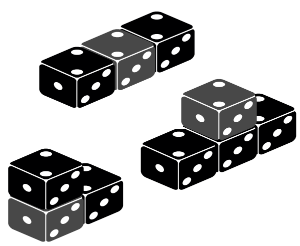
which holds when $n \geq 45$. I.e. $p_{46} < p_{45}$ and the sequence goes decrescendo from there on. The maximal portion is thus the one of guest number 45.

Problem 2 (Alea iacta est)**.** Let Δ represent the difference between the largest possible sum and the smallest possible sum of all visible faces on a dice configuration. Imagine a construction like the one below but where the number of 'holes' is not 5 but some larger number g. If $\Delta = 2032$ for that construction, what is the number of holes (g)?



Answer. 201

Solution. Note that Δ is additive: the total value is the sum of the Δ for each dice. The contribution to Δ of a dice depends on its position.



(i) If a dice is attached to two other dice along opposite faces (we'll call those linear dice), the sum of visible sides is 14 independently of their orientation. Hence $\Delta = 0$ for each of those dice. (ii) If a dice is located on a corner (there are 4 of those), two adjacent sides are covered. The maximal amount covered is $11 = 6 + 5$ and the minimal amount covered is $1 + 2 = 3$. Therefore Δ for those dice is $(21 - 3) - (21 - 11) = 8$. (iii) Finally, dice located at a T junction have two opposite sides covered (whose pips add up to 7) and an extra side. The amount hidden varies thus between $8 = 7 + 1$ and $13 = 7 + 6$. For those dice, Δ is thus equal $(21 - 8) - (21 - 13) = 5$.
If we let L be the number of linear dice, C the number of corner dice and T the number of T-junction dice, we can write

$$\Delta = 8C + 5T.$$

Note that there are four corners $C = 4$ and in our case $\Delta = 2032$ so that $5T = 2000$ and $T = 400$. Between any two adjacent holes, there are 2 junction dice. Hence $g - 1 = \frac{T}{2}$ and here $g = 201$.

Problem 3 (This problem stinks)**.** The septic number system consists of the positive integers of the form $7n+1$: that is, 1, 8, 15, 22, etc. A septic prime is a septic number larger than 1 that cannot be written as a product of two smaller septic numbers. Every septic number larger than 1 can be written as a product of septic primes, but this factorization is not always unique. For example, $36 \times 169 = 78 \times 78$, and all of 36, 169, and 78 are septic primes. In this instance our two factorizations have length 2, where the length is the number of septic primes involved in the factorization (with repeated primes counted multiply).

For each septic integer n, let

$$E(n) = \frac{\text{largest length of a factorization of } n \text{ into septic primes}}{\text{smallest length of such a factorization}}.$$

Find the largest possible value of $E(n)$.

Answer. 3

Solution. We first argue that a value at least 3 is possible. Let $A = 3^6$ and $B = 5^6$. By Fermat's little theorem, $A \equiv B \equiv 1 \pmod 7$, and so A and B are septic numbers. We claim they are both septic primes. Any nontrivial factorization of A in the positive integers has the form $3^{e_1} \cdot 3^{e_2}$, with e_1, e_2 positive integers adding to 6. Since 3^e is not 1 mod 7 for any positive integer $e < 6$, none of those factorizations are valid septically. Thus, A is a septic prime. A parallel argument shows B is a septic prime. Now notice that

$$A \cdot B = 15 \cdot 15 \cdot 15 \cdot 15 \cdot 15 \cdot 15$$

and that 15 is a septic prime (since its nontrivial factors, 3 and 5, are not 1 mod 7). Thus, AB has a factorization as a product of 2 septic primes and as a product of 6 septic primes, and so $E(AB) \geq 6/2 = 3$.

Next we prove that no value larger than 3 is possible. It is helpful to separate out from the main proof the following key observation.

Lemma 1. Every septic prime is a product of at most six ordinary primes.

Proof. Let P be a septic prime and suppose for a contradiction that $P = p_1 \cdots p_m$, where each p_i is an ordinary prime and $m > 6$.

Consider the list of $m-1$ numbers p_1, p_1p_2, $p_1p_2p_3$, $\ldots$, $p_1\cdots p_{m-1}$. Since P is an integer multiple of each of number on this list, and P is not a multiple of 7, no term in the list is congruent to 0 mod 7. Furthermore, no term is congruent to 1 mod 7. To see this, suppose some number R on this list is congruent to 1 mod 7, and let $S = P/R$. Since $P = SR$ and both P and R are congruent to 1 mod 7, it must be that $S \equiv 1 \pmod 7$ also. But then R, S are septic numbers larger than 1 with $P = RS$, contradicting that P is a septic prime. Putting these observations together, we conclude that each term on our list belongs to one of the 5 residue classes $2, 3, 4, 5, 6 \pmod 7$.

Since the list has length $m-1 > 6-1 = 5$, two terms must coincide mod 7; say $p_1\cdots p_k \equiv p_1\cdots p_\ell \pmod 7$, where $1 \le k < \ell \le m-1$. Canceling, $p_{k+1}\cdots p_\ell \equiv 1 \pmod 7$. (We use here that 7 is prime, so that cancellation is valid in the integers modulo 7.) Now taking $R = p_{k+1}\cdots p_\ell$ and $S = P/R$, we get the same contradiction as before: R and S are septic numbers larger than 1 multiplying to P. $\square$

Now n be a septic integer larger than 1 and suppose we have two factorizations of n into septic primes p_i and q_j, say

$$n = p_1\cdots p_k = q_1\cdots q_\ell, \tag{*}$$

where $k \ge \ell$. We must show that $k/\ell \le 3$. Let us assume to start with that no p_i or q_j is prime in the ordinary integers.

In this case, we can show that the ratio $k/\ell \le 3$ by counting the number of ordinary prime factors of n. We count with multiplicity, meaning that if a prime appears to the rth power in n, it is counted r times. Since each q_j has at most 6 ordinary prime factors (by the lemma), $n = q_1\cdots q_\ell$ has at most 6ℓ ordinary prime factors. On the other hand, since no p_i is prime, each p_i has at least two ordinary prime factors. Thus, $n = p_1\cdots p_k$ has at least $2k$ ordinary prime factors. Hence,

$$2k \le \#\text{ ordinary prime factors of } n \le 6\ell,$$

forcing $k/\ell \le 6/2 = 3$.

Now suppose that some p_i is prime, say p_1. Then by unique factorization in the ordinary integers, some $q_j = p_1$, and we can assume (reordering if necessary) that $j = 1$. Canceling $p_1 = q_1$ in (*),

$$p_2\cdots p_k = q_2\cdots q_\ell.$$

If any remaining p_i is an ordinary prime, we can continue the process. After finitely many steps, we are left with a factorization where none of the remaining p_i or q_j are prime. (If a remaining q_j were an ordinary prime, it would have to equal some p_i, but we removed all p_i that are ordinary primes.) If we have removed s primes, then either $s = k = \ell$ — which forces $\frac{k}{\ell} = 1$ — or from the case handled in the last paragraph,

$$\frac{k-s}{\ell-s} \leq 3.$$

But $\frac{k}{\ell} \leq \frac{k-s}{\ell-s}$ (since $\ell \leq k$), and so $\frac{k}{\ell} \leq 3$ in this case as well.

Authors.