Sponsored by: UGA Math Department and UGA Math Club

Team Round / 1 hour / 210 points
October 26, 2019

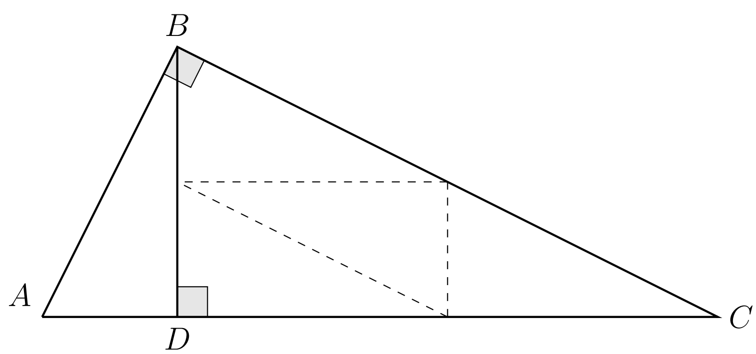
**WITH SOLUTIONS**

**Problem 1** (Cold-blooded mathematics)**.** Recall that an object is an $n$-*reptile* if it can be decomposed into $n$ congruent pieces each similar to the original figure.

If a right triangle with shortest leg 1 is a 5-reptile, what is the length of the hypotenuse?

**Answer.** $\sqrt{5}$

**Solution.** To find such a decomposition, drop a perpendicular from the vertex of the right triangle to the hypotenuse.




Then $\Delta ABD$, $\Delta BCD$, and $\Delta ACB$ are similar. Now decompose $\Delta BCD$ as a 4-reptile (every triangle is a 4-reptile!) by joining the midpoints of the sides (dashed lines).

The 5 interior triangles are now all similar to $\Delta ABC$, and to each other. To guarantee that they are all congruent, we just need to make sure that they all have the same area, meaning $\text{Area}(\Delta ABC) = 5 \cdot \text{Area}(\Delta ABD)$. Since area transforms as the square of the similarity ratio, the sides of $\Delta ABC$ are $\sqrt{5}$ times as long as the corresponding sides of $\Delta ABD$. In particular, the hypotenuse has length $\sqrt{5}$.

Thus, the answer is $\sqrt{5}$ — as long as there is a unique answer ! And there is: With a bit of experimentation, you can convince yourself that the above construction is essentially unique. In particular, any 5-reptile right triangle has side lengths in a ratio $1 : 2 : \sqrt{5}$.

*Remark.* Reptiles (usually with the spelling "rep-tiles") were first studied by Solomon Golomb and were popularized by Martin Gardner in the Mathematical Games column in the May 1963 issue of *Scientific American.*

**Problem 2** (Colors and numbers)**.** If the positive integers from 1 to 30 are all colored the same color, then there are guaranteed to be numbers $x, y, z$ that are all the same color and satisfy $x+y=z$ — a "monochromatic solution to $x + y = z$". At the other extreme, if the positive integers from 1 to 30 are colored 30 different colors, then there are no monochromatic solutions to $x + y = z$. What is the smallest integer $n$ for which it is possible to color 1 to 30 with $n$ colors and have no monochromatic solution to $x + y = z$?

**Note:** We do *not* require that $x, y, z$ be distinct. That is, a solution to $x + y = z$, where $x = y$, and where $x$ and $z$ share the same color, counts as a monochromatic solution.

**Answer.** 4 (colors)

**Solution.** The answer is at most 4, as shown by the following coloring with colors $A, B, C, D$:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $1_A$ | $2_B$ | $3_B$ | $4_A$ | $5_C$ | $6_C$ | $7_C$ | $8_C$ | $9_C$ | $10_A$ |
| $11_D$ | $12_D$ | $13_D$ | $14_D$ | $15_D$ | $16_D$ | $17_D$ | $18_D$ | $19_D$ | $20_D$ |
| $21_A$ | $22_B$ | $23_B$ | $24_A$ | $25_C$ | $26_C$ | $27_C$ | $28_C$ | $29_C$ | $30_A$ |

To prove that the answer is at least 4, we must show that there is no way to color $1, 2, \ldots, 30$ with three colors that avoids a monochromatic solution.

Suppose that we can color $1, 2, 3, \ldots, N$ with colors $A, B, C$ avoiding a monochromic solution to $x + y = z$. Let $X$ be the number of elements receiving the most popular color, say (without loss of generality) color $A$. By the Pigeonhole principle,

$$N \leq 3X.$$

List the integers having color $A$,

$$n_1 < n_2 < \cdots < n_X,$$

and consider the $X - 1$ differences

$$n_2 - n_1, \quad n_3 - n_1, \quad \ldots, \quad n_X - n_1. \tag{*}$$

None of these differences can have color $A$: If $n_k - n_1$ has color $A$, then $n_k = n_1 + (n_k - n_1)$ is an $A$-monochromatic solution to $x + y = z$, contrary to assumption. So at least half of the numbers (*) share the same color, say $B$; list these numbers as

$$m_1 < m_2 < \cdots < m_Y,$$

where

$$X - 1 \leq 2Y.$$

Now consider the $Y - 1$ differences

$$m_2 - m_1, \quad m_3 - m_1, \quad \ldots, \quad m_Y - m_1. \tag{**}$$

None of these can have color $B$, as otherwise we have (as above) a $B$-monochromatic solution to $x + y = z$. But none can have color $A$ either: Each $m_k - m_1$ is a difference of two numbers from (*), and so has the form $n_j - n_i$ for some $i$ and $j$. If $n_j - n_i$ has color $A$, then $n_j = n_i + (n_j - n_i)$ is an $A$-monochromatic solution to $x + y = z$. Thus, the $Y - 1$ numbers (**) have color $C$.

If $Y - 1 \geq 2$, consider the difference between any two numbers in the list (**). Arguing as above, we find that this difference cannot have color $C$, $B$, or $A$, which is absurd! So

$$Y \leq 2.$$

But then

$$N \leq 3X \leq 3(X - 1) + 3 \leq 3(2Y) + 3 \leq 15.$$

So as soon as $N > 15$, any coloring of 1 through $N$ using 3 colors is guaranteed to have a monochromatic solution to $x + y = z$. In particular, this is guaranteed when $N = 30$.

*Remark.* Let $N = N(k)$ be the largest positive integer for which it is possible to color $1, 2, \ldots, N$ with $k$ colors and avoid a monochromatic solution to $x + y = z$. We have seen above that $N(3) \leq 15$. Issai Schur proved in 1917 that $N(k)$ exists for all positive integers $k$, and in fact that

$$N(k) < \lfloor ek! \rfloor,$$

where $e = 2.71828\ldots$ is the base of the natural logarithm. Our proof that $N(3) \leq 15$ was obtained by specializing Schur's argument to $k = 3$. We invite you to do the reverse: generalize our proof to establish the inequality for $N(k)$ given above.

It has been shown that $N(1) = 1, N(2) = 4, N(3) = 13, N(4) = 44$, and $N(5) = 160$. The values of $N(k)$ are unknown for $k > 5$. That $N(5) = 160$ was not established until 2018; the computer-assisted proof of this fact occupies more than two petabytes!

**Problem 3** (Unscrambling an egg)**.** The average of a set of integers is computed by taking the sum of the elements divided by the total number of elements. For example, the average of the set $\{1, 5\}$ is $\frac{1+5}{2} = 3$ and the average of the set $\{1, 5, 6\}$ is $\frac{1+5+6}{3} = 4$.

Let $A$ be a set with 7 elements (so $A$ has 127 nonempty subsets). The averages of all of the 127 subsets of $A$ are listed below, in increasing order. What are the 7 elements of $A$?

Write the numbers you find in increasing order. *You must have all the numbers correct to receive credit for this problem.*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 759 | 27 | 4014 | 53 | 5043 | 79 | 5659 | 105 | 6723 |
| 2 | 969 | 28 | 4119 | 54 | 5064 | 80 | 5694 | 106 | 6744 |
| 3 | 1179 | 29 | 4119 | 55 | 5099 | 81 | 5799 | 107 | 6779 |
| 4 | 1319 | 30 | 4224 | 56 | 5127 | 82 | 5799 | 108 | 6807 |
| 5 | 1389 | 31 | 4259 | 57 | 5169 | 83 | 5799 | 109 | 6919 |
| 6 | 1599 | 32 | 4259 | 58 | 5169 | 84 | 5883 | 110 | 6975 |
| 7 | 2019 | 33 | 4287 | 59 | 5169 | 85 | 5904 | 111 | 7059 |
| 8 | 2334 | 34 | 4329 | 60 | 5211 | 86 | 5939 | 112 | 7164 |
| 9 | 2439 | 35 | 4329 | 61 | 5211 | 87 | 6009 | 113 | 7199 |
| 10 | 2719 | 36 | 4371 | 62 | 5239 | 88 | 6009 | 114 | 7374 |
| 11 | 2859 | 37 | 4399 | 63 | 5239 | 89 | 6009 | 115 | 7479 |
| 12 | 2964 | 38 | 4434 | 64 | 5259 | 90 | 6009 | 116 | 7619 |
| 13 | 3069 | 39 | 4469 | 65 | 5274 | 91 | 6051 | 117 | 7689 |
| 14 | 3279 | 40 | 4539 | 66 | 5295 | 92 | 6079 | 118 | 7759 |
| 15 | 3279 | 41 | 4539 | 67 | 5379 | 93 | 6114 | 119 | 7899 |
| 16 | 3384 | 42 | 4539 | 68 | 5379 | 94 | 6135 | 120 | 8214 |
| 17 | 3447 | 43 | 4539 | 69 | 5379 | 95 | 6219 | 121 | 8319 |
| 18 | 3489 | 44 | 4644 | 70 | 5379 | 96 | 6219 | 122 | 8739 |
| 19 | 3559 | 45 | 4679 | 71 | 5379 | 97 | 6219 | 123 | 8949 |
| 20 | 3699 | 46 | 4707 | 72 | 5379 | 98 | 6324 | 124 | 9159 |
| 21 | 3699 | 47 | 4819 | 73 | 5379 | 99 | 6359 | 125 | 9579 |
| 22 | 3783 | 48 | 4819 | 74 | 5519 | 100 | 6429 | 126 | 9789 |
| 23 | 3804 | 49 | 4854 | 75 | 5547 | 101 | 6499 | 127 | 9999 |
| 24 | 3839 | 50 | 4959 | 76 | 5589 | 102 | 6534 | | |
| 25 | 3867 | 51 | 4959 | 77 | 5589 | 103 | 6639 | | |
| 26 | 3979 | 52 | 4959 | 78 | 5631 | 104 | 6639 | | |

**Answer.** 759, 1179, 2019, 5379, 7899, 9579, 9999

**Solution.** There are many ad hoc approaches to this problem. In the following we will outline a computationally efficient method valid for any finite starting set $A$.

Since $A$ is a set, all the $a_i$ are distinct, and we can list them in increasing order as $a_1 < a_2 < \cdots < a_n$. (Distinctness is not actually necessary for the argument, but makes some steps cleaner.) All the $a_i$ appear in the list because they are the averages of the 1-element subsets. In particular the smallest average must be $a_1$. Now suppose we have determined the $k$ smallest elements $a_1, \ldots, a_k$. Then we classify each average in our list as "known" if it comes from some subset of $\{a_1, \ldots, a_k\}$ or "unknown" if it does not (if an average repeats in our list, then it's possible some occurrences are known and others are unknown).

Let $x$ be the smallest unknown average. If $x$ arises as an average of $m+1$ elements of $A$ then it must have the form

$$x = \frac{a_{k+1} + \sum_{i=1}^{m} a_i}{m+1},$$

as any other average of $m+1$ elements is either larger or known. Notice that $1 \leq m \leq k$.

From here we can use a little trial and error to determine the value of $a_{k+1}$ in our particular case (and defer until the end an argument which removes the reliance on trial and error).

First we know $a_1 = 759$, so our smallest unknown average is 969, and since $1 \leq m \leq k$, we get $m = 1$. This gives the equation

$$969 = \frac{a_2 + a_1}{1+1},$$

which implies $a_2 = 1179$.

Now that we know $a_1$ and $a_2$, our smallest unknown average is 1319. Since $1 \leq m \leq k$ we have two possibilities:

$$1319 = \frac{a_3 + a_1}{1+1},$$

or

$$1319 = \frac{a_3 + a_1 + a_2}{2+1},$$

The first of these gives $a_3 = 1879$ which is not possible, because 1879 does not appear in our list of averages. On the other hand, the second equation gives $a_3 = 2019$, which does appear in the list of averages.

We could continue this process to find all the numbers, but this will get computationally expensive. Instead, now we will switch to finding the largest 3 elements of $A$.

In exact analogy to before, we find $a_7 = 2019$, and $a_6$ satisfies

$$9789 = \frac{a_6 + a_7}{2},$$

giving $a_6 = 9579$. The largest unknown average is 9159, so either

$$9159 = \frac{a_5 + a_7}{2},$$

or

$$9159 = \frac{a_5 + a_6 + a_7}{3}.$$

The first of these solves to $a_5 = 8319$, while the second solves to $a_5 = 7899$. Both seem initially possible, but we can note that using 8319 in averages gives rise to numbers which don't appear. For example $\frac{8319+9579}{2} = 9054$ which does not appear. Thus $a_5 = 7899$.

We have determined all elements except $a_4$. We could of course proceed as before with more cases for our trial and error, but instead we will give an argument that actually the average of $A$ itself is the middle average on the list, which gives the identity

$$5259 = \frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7}{7},$$

which solves to give $a_4 = 5379$.

*Lemma.* The average of $A$ is the middle average on the list.

**Proof.** For convenience we use the notation $|X|$ to denote the size of the set $X$. We shall also use $X^C$ to denote the complement of $X$ in $A$, that is $X^C$ contains all the elements in $A$ that are not in $X$. Finally, we let $\text{AVG}_S$ be the average of set $S$.

Notice that

$$|X| \cdot \text{AVG}_X + |X^C| \cdot \text{AVG}_{X^C} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7.$$

Dividing by 7 we find

$$\frac{|X| \cdot \text{AVG}_X + |X^C| \cdot \text{AVG}_{X^C}}{7} = \text{AVG}_A.$$

Hence, the average of $A$ can be written as a weighted average of the averages of $|X|$ and $|X|^C$. This means $\text{AVG}_A$ lies between $\text{AVG}_X$ and $\text{AVG}_{X^C}$ for each $X$ and so $\text{AVG}_A$ is in the middle of the list of averages (for us this is the 64th average).

Q.E.D.

We now explain how to avoid the guess and check strategy of above.

Define

$$A_p := \frac{a_{k+1} + \sum_{i=1}^{p} a_i}{p+1}$$

Then if $x = A_m$ is the smallest unknown average, it satisfies $x \leq A_{m+1}$ and $x \leq A_{m-1}$.

Note that

$$\begin{aligned}(m+1)x + a_{m+1} &= (m+2)A_{m+1} \\ &\geq (m+2)x\end{aligned}$$

Rearranging this implies $x \leq a_{m+1}$.

Similarly, note that

$$\begin{aligned}(m+1)x - a_m &= mA_{m-1} \\ &\geq mx.\end{aligned}$$

Rearranging, this implies $a_m \leq x$.

Therefore $a_m \leq x \leq a_{m+1}$, so we can determine $m$ (and therefore $a_{k+1}$) by examining the position of $x$ relative to our already known numbers $a_1, \ldots, a_k$.

Using this process on the given list of averages, we first find $a_1 = 759$ and the smallest unknown average is 969. This average appears after $a_1$, so $m = 1$ and $a_2$ satisfies

$$969 = \frac{a_2 + a_1}{2},$$

so $a_2 = 1179$.

Now the smallest unknown average is 1319, which appears after $a_2$, so now $m = 2$ and $a_3$ satisfies

$$1319 = \frac{a_3 + a_1 + a_2}{3},$$

so $a_3 = 2019$.

Now we have to compute the new known averages, and once we do, we find 2334 is the smallest unknown average, so now $m = 3$ and $a_4$ satisfies

$$2334 = \frac{a_4 + a_1 + a_2 + a_3}{4},$$

so $a_4 = 5379$.

Now we are pushing the boundaries of what is reasonable to compute by hand, but if you continued with the process, you would find 2964 is the smallest unknown average, which appears after $a_3$ but before $a_4$, so $m = 3$ and $a_5$ satisfies

$$2964 = \frac{a_5 + a_1 + a_2 + a_3}{4},$$

so $a_5 = 7899$.

Continuing, the next smallest unknown you get is 3384 which again appears between $a_3$ and $a_4$, so $m = 3$ and $a_6$ satisfies

$$3384 = \frac{a_6 + a_1 + a_2 + a_3}{4},$$

so $a_6 = 9579$.

Finally, the last smallest unknown average is 3489 which also appears between $a_3$ and $a_4$, so $m = 3$ and $a_7$ satisfies

$$3489 = \frac{a_7 + a_1 + a_2 + a_3}{4},$$

so $a_7 = 9999$.

*Note.* You might be wondering why so many numbers above end with 9s. The idea is that the authors wanted all the averages to be integers, which was ensured by making sure all the starting numbers were the same modulo 2, 3, 4, 5, 6, and 7. This comes out to be equivalent to a modulo 420 condition, and so lots of the averages have the same last digit as the chosen starting numbers.

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