1785

Sponsored by: UGA Math Department and UGA Math Club

Ciphering Round / 2 minutes per problem
October 20, 2018

WITH SOLUTIONS

Problem 1. If x is twice the square of half of the square root of 2018, what is $x+1$ (simplified)?

Answer. 1010

Solution. Notice that

$$x = 2\left(\frac{1}{2}\sqrt{2018}\right)^2 = 2\left(\frac{1}{4}\cdot 2018\right) = \frac{1}{2}\cdot 2018 = 1009,$$

so $x+1 = 1010$.

Problem 2. The area of a circle is 100π units^2. If the radius is increased by 1 unit, how much will the area change?

Answer. 21π units^2

Solution. If $A = 100\pi = \pi r^2$, then $r = 10$. Increase r by 1 to get the new area $\pi\left(11^2\right) = 121\pi$. Thus the area increased by 21π.

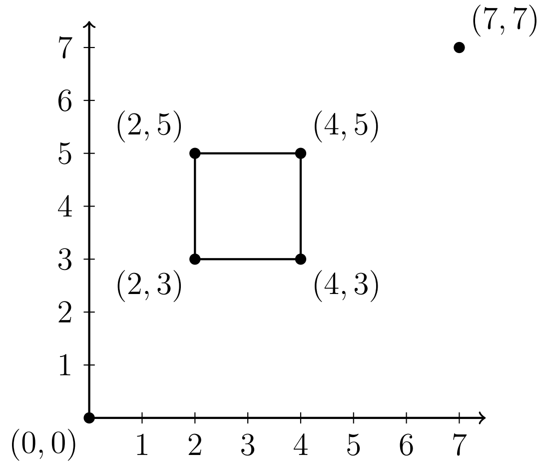
Note. It is not a coincidence that this is very close to the circumference $2\pi r = 20\pi$ of the original circle !

Problem 3. If $x + \frac{1}{x} = 3$, what is $x^2 + \frac{1}{x^2}$?

Answer. 7

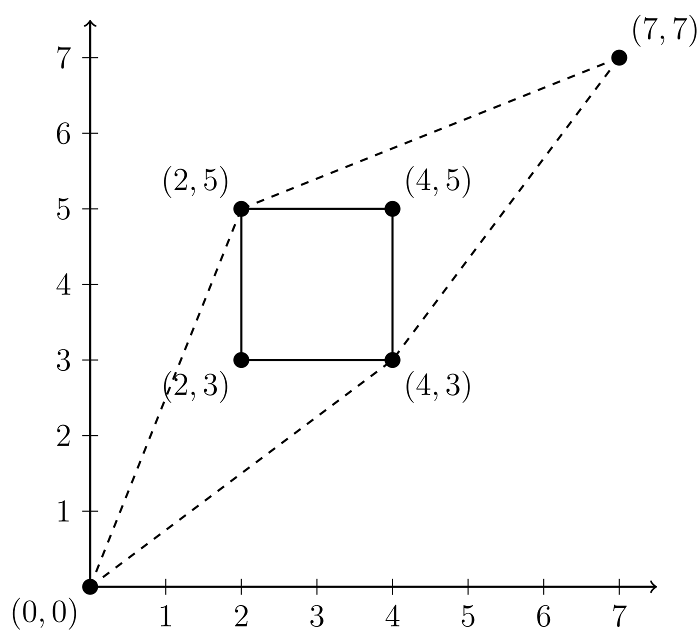
Solution. Square both sides of $x + \frac{1}{x} = 3$ to get $x^2 + 2 + \frac{1}{x^2} = 9$. Thus $x^2 + \frac{1}{x^2} = 7$.

Problem 4. What is the length of the shortest path from $(0, 0)$ to $(7, 7)$ that does not go inside the square shown? The path may touch the square.



Answer. 10 units

Solution. There are two candidates for shortest path: $(0, 0) \to (4, 3) \to (7, 7)$ and $(0, 0) \to (2, 5) \to (7, 7)$. The first has length 5 on each segment while the second has length $\sqrt{29}$ on each segment. Since $5 < \sqrt{29}$, the former gives the shortest path.



Problem 5. Find the nearest integer to $1024^{\log_2(256)} - 256^{\log_2(1024)}$.

Answer. 0

Solution. Observe that

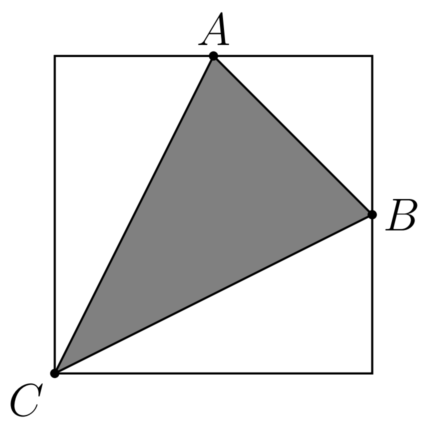
$$1024^{\log_2(2^8)} - 256^{\log_2(2^{10})} = 1024^8 - 256^{10} = \left(2^{10}\right)^8 - \left(2^8\right)^{10} = 0.$$

Note. In fact, for any x, $y > 0$ and for any base b

$$x^{\log_b(y)} = y^{\log_b(x)}.$$

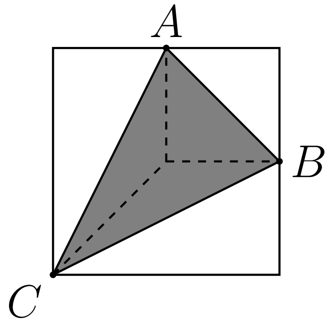
You can check this by taking $\log_b$ of both sides and using properties of logarithms.

Problem 6. If A and B are midpoints of the sides in the 1 by 1 square shown, what is the area of the shaded region?



Answer. 3/8

Solution. Two of the unshaded triangles have area $\frac{1}{2}(1)(\frac{1}{2}) = \frac{1}{4}$, while the third unshaded triangle has area $\frac{1}{2}(\frac{1}{2})(\frac{1}{2}) = \frac{1}{8}$, so the total *shaded* area is $1-2\left(\frac{1}{4}\right)-\frac{1}{8} = \frac{3}{8}$. Alternatively, we can directly break the shaded triangle into three subtriangles each with base and height $\frac{1}{2}$:

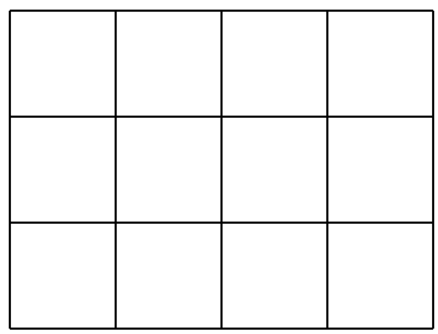


Problem 7. What is the largest prime factor of the binomial coefficient $\binom{100}{50}$?

Answer. 97

Solution. Recall that $\binom{100}{50} = \frac{100!}{50!\,50!}$, so any prime factor of $\binom{100}{50}$ must be a prime factor of 100!. Since 97 is the largest prime less than 100 (so it divides 100!) and 97 is not a factor of 50!, it is the answer.

Problem 8. How many rectangles are in the grid below? Only count those rectangles whose edges lie on the lines shown.



Answer. 60

Solution. A rectangle is determined by the two vertical lines and the two horizontal lines on which its edges lie. There are $\binom{5}{2} = 10$ ways to choose two vertical lines and $\binom{4}{2} = 6$ ways to choose two horizontal lines, so there are 60 total rectangles.

Problem 9. The positive integers a and b each have exactly two prime factors: 2 and 3. If a does not divide b and b does not divide a, what is the smallest that a can be?

Answer. 12

Solution. Both a and b are of the form $2^n 3^m$ where n and m are at least 1. Thus 6 divides a. If $a = 6$ then a will divide b. The next smallest option for a is $2 \cdot 6 = 12$, and the pair $\{a, b\} = \{12, 18\}$ satisfies the conditions.

Note. In fact $\{a, b\}$ is a satisfactory pair if and only if it takes the form $\{2^n 3^m, 2^x 3^y\}$ where $1 \leq n < x$ and $1 \leq y < m$.

Problem 10. What is the first row of Pascal's triangle that has an entry larger than 2018? The n^{th} row is the one that begins $1, n, \ldots$.

Answer. the 14th row

Solution. The first occurence will happen in the central binomial coeficient $\binom{2n}{n}$ or $\binom{2n+1}{n}$. We can work from an initial guess of say $\binom{10}{5}$:

$$\binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2 \cdot 3 \cdot 7 \cdot 6 = 252,$$

$$\binom{11}{5} = \frac{11}{6}\binom{10}{5} \approx 2 \cdot 250 = 500,$$

$$\binom{12}{6} = \frac{12}{6}\binom{11}{5} \approx 2 \cdot 500 = 1000,$$

$$\binom{13}{6} = \frac{13}{7}\binom{12}{6} \approx 2 \cdot 1000 = 2000,$$

$$\binom{14}{7} = \frac{14}{7}\binom{13}{7} \approx 2 \cdot 2000 = 4000.$$

Checking

$$\binom{13}{6} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 13 \cdot 11 \cdot 3 \cdot 4 = (12+1)(12-1)12 < 12^3 = 1728,$$

tells us the true answer is the 14^{th} row.

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