Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 22, 2022

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

   where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit quietly.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. Suppose a, b, c, and d are nonzero digits 1, 2, ..., 9 and

$$ab^c - d = 2022.$$

What is $a+b+c+d$? Note: Here ab is a two digit number, not a product.

(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

Problem 2. Let a and b be nonnegative integers such that $ab < a^b < a+b$. To which of the following is $(a+b)^2$ equivalent.

(A) a^2 (B) b^2 (C) ab (D) $4ab$ (E) $a+b$

Problem 3. You can write

$$x^{1022} = (x^2-1)f(x) + Ax + B$$

for some polynomial $f(x)$, where A and B are real numbers. What is A^2+B^2?

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) $\frac{5}{2}$

Problem 4. Let $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + x^n$ be the monic polynomial of the lowest degree vanishing exactly at the positive divisors of 16 (i.e. $p(m) = 0$ if and only if $m \geq 0$ and m divides 16). What is $p(0)$?

(A) -31 (B) 31 (C) 256 (D) -1024 (E) 1024

Problem 5. Suppose you have 100 positive integers with the property that

$$a_1 < a_2 < \cdots < a_{99} < A < a_{100},$$

where A is the average of the 100 numbers $a_1, \ldots, a_{100}$. What is the smallest that a_{100} can be?

(A) 100 (B) 4950 (C) 4951 (D) 5050 (E) This is not possible

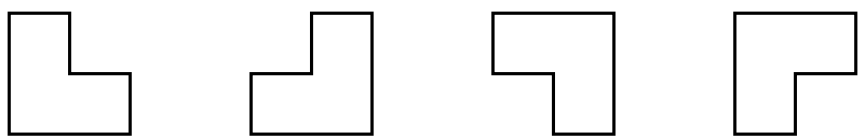
Problem 6. The average of n numbers $a_1, a_2, \ldots, a_n$ is 10. If each a_k is increased by k, the n resulting numbers have an average of 20. What is n?

(A) 4 (B) 10 (C) 19 (D) 20 (E) There is not a unique solution.

Problem 7. We know that there are $8! = 40320$ permutations of the set $\{1,2,3,4,5,6,7,8\}$. How many of those permutations map the small numbers $\{1,2,3,4\}$ to the large numbers $\{5,6,7,8\}$?

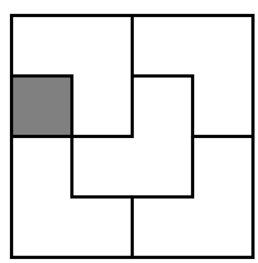
(A) 24 (B) 576 (C) 2520 (D) 10080 (E) 20160

Problem 8. An *L-tromino* is a shape made by gluing three unit squares into the shape of an L. Take a 4×4 square board, consisting of 16 unit squares, and choose a unit square. It is always possible to cover the entire board except for the chosen square with L-trominos, rotating them if necessary. In such a tiling, each L-tromino is in one of four different orientations:

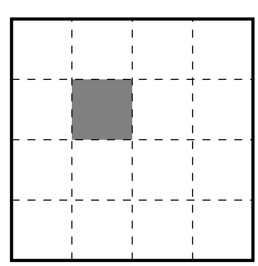


Orientation 1 Orientation 2 Orientation 3 Orientation 4

For a tiling of the board minus a square, define a_1 be the number of L-trominos in Orientation 1 (as defined above), and similarly define a_2, a_3, a_4. For example, in the tiling below, we have $a_1 = 1$, $a_2 = a_3 = 2$, $a_4 = 0$.



Consider now a tiling of the board minus the shaded square shown below:



Which of the following is the largest?

(A) a_1 (B) a_2 (C) a_3 (D) a_4 (E) Cannot be determined

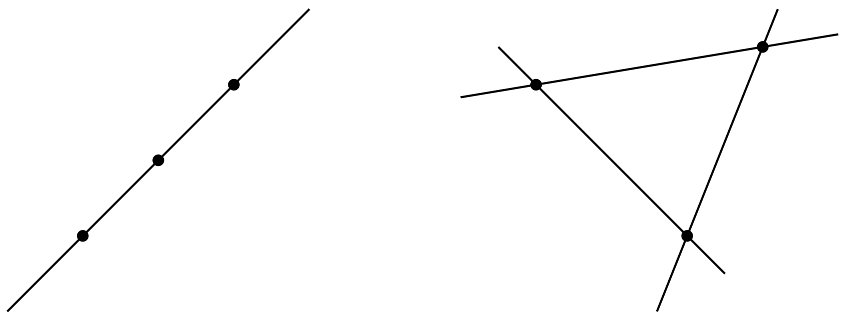
Problem 9. Find a digit y, that is $y \in \{0, 1, \ldots, 9\}$, so that for *every* digit x, the 10 digit number

$$1022xy2022$$

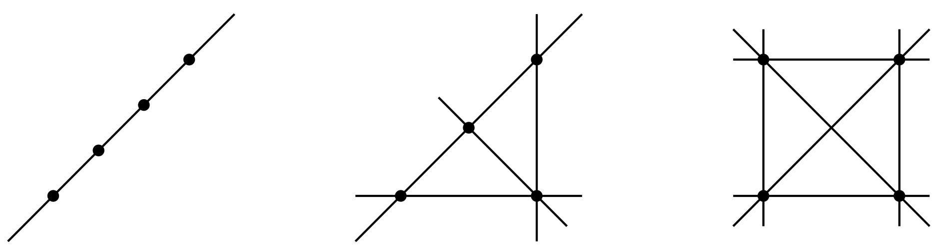
is not divisible by 11.

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Problem 10. We all know two distinct points in the plane determine one line, while three distinct points determine either one or three lines; depending on whether the points are colinear:



So we say there are 2 different configurations of 3 points.
Similarly, there are 3 different configurations of 4 points:



How many different configurations of 5 points are there?

(A) 4 (B) 5 (C) 6 (D) 8 (E) 9

Problem 11. How many rectangles are there in a 2022×2022 chessboard? Yes, one of the listed options is the correct answer!

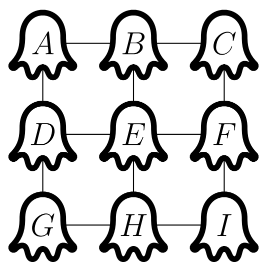
(A) 4183059834002 (B) 4183059834003 (C) 4183059834007 (D) 4183059834008 (E) 4183059834009

Problem 12. Assume (x, y, z) is a Pythagorean triple, that is $x^2 + y^2 = z^2$. Assume x and z are successive odd numbers and x is not divisible by 3. Which of the following is guaranteed to be a divisor of y?

(A) 9 (B) 10 (C) 12 (D) 15 (E) 21

The following description will be used for the next three problems. Beware: they're squary! Define a *spooky square* with *spooky sum* S to be an arrangement of the numbers 1 through 9 in the nine spaces below such that each of the four 2×2 subsquares adds to the same number S. That is,

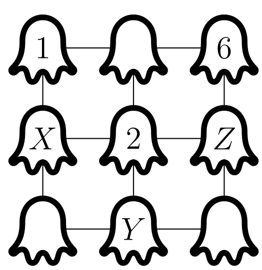
$$S = A + B + D + E = B + C + E + F = D + E + G + H = E + F + H + I.$$



Problem 13. Which of the following is equivalent to $C + G - A$ for all spooky squares?

(A) A (B) $3E - F - H$ (C) $B + D - E$ (D) $F + H - E$ (E) I

Problem 14. Complete the spooky square below. What is $X + Y + Z$?

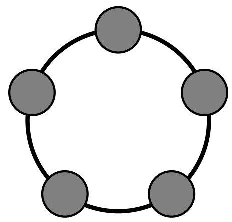


(A) 16 (B) 17 (C) 18 (D) 19 (E) 20

Problem 15. There are 376 distinct spooky squares. What is the average of all 376 spooky square sums?

(A) 17 (B) 18 (C) 19 (D) 20 (E) 21

Problem 16. Five lamps are on a circular circuit, as shown below. Manually toggling a lamp, i.e., switching it from off to on or vice versa, automatically toggles the two lamps adjacent to it.



Initially, all lamps are switched off. What is the least number of moves (i.e., manual toggles) required to have them all switched on at the same time?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 17. In how many ways can you write 100 as an unordered sum of 1's, 2's, and 3's. Unordered here means $1+2$ counts the same as $2+1$ and similarly for other rearrangements of sums.

(A) 520 (B) 884 (C) 948 (D) 1020 (E) 1326

Problem 18. The number S given by the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{1}{10^n - 1} = \frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \dots$$

has decimal expansion

$$S = 0.d_1 d_2 d_3 \cdots = \sum_{k=1}^{\infty} d_k 10^{-k}$$

where d_k is the kth decimal digit after the decimal point. Find d_{22}.

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 19. The nonzero digits A, B, C, D are such that the two-digit number AB divides the two-digit number CD, and the four-digit number $ABCD$ is a perfect square. Find $A+B+C+D$.

(A) 9 (B) 12 (C) 16 (D) 18 (E) 25

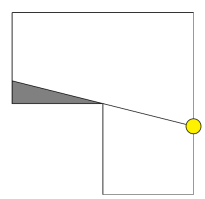
Problem 20. Leda squares a positive integer and observes that its first (that is, leftmost) X digits are all the same (and nonzero). What is the largest possible value of X?

(A) 2 (B) 3 (C) 4 (D) 5 (E) There is no largest

Problem 21. Edna squares a positive integer and observes that its last (that is, rightmost) X digits are all the same (and nonzero). What is the largest possible value of X?

(A) 2 (B) 3 (C) 4 (D) 5 (E) There is no largest

Problem 22. Consider an L shaped room made of three square parts arranged as in the picture below. If the walls do not reflect light, what is the expected proportion of the room which is lit when a bulb is placed randomly inside the room. In the example illustrated below, the bulb illuminates $\frac{23}{24}$ of the room.



(A) $\frac{5}{6}$ (B) $\frac{8}{9}$ (C) $\frac{11}{12}$ (D) $\frac{14}{15}$ (E) $\frac{17}{18}$

Problem 23. Anna has defined a new operation, which she calls $\star$, on pairs of real numbers: if $P = (x, y)$ and $Q = (x', y')$ then she lets $P \star Q = (x'', y'')$ as

$$\begin{cases} x'' &= xx' - yy' \\ y'' &= xy' + x'y \end{cases}.$$

Lisa picks R to be the pair $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and tells Anna, "I know what the value of

$$\underbrace{R \star \cdots \star R}_{\text{2022 copies of R}}$$

is!" Can you find the result of Lisa's computation?

(A) $(-1, 0)$ (B) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (C) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ (D) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (E) $(1, 0)$

Problem 24. How many positive integers n with $1 \leq n \leq 2022$ are there that **cannot** be written in the form $n = k + \lfloor \log_2 k \rfloor$ for some k? Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(A) 9 (B) 10 (C) 11 (D) 12 (E) 13

Problem 25. A magic square is a grid with numbers such that the sum of all entries in any given row, column or diagonal is a fixed constant.

Consider a magic square, all whose entries are strictly positive integers (but not necessarily distinct), and of which only one of the entries is known:

X		
	9	

What is the minimal possible value of the entry in the upper left corner?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5