1785

Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 26, 2019

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. How many of the following equations have solutions in positive integers?

i) $28x + 30y + 31z = 365$
ii) $29x + 30y + 31z = 366$
iii) $10x + 26y = 2019$
iv) $3x + 4y = 5z$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 2. Suppose that a, b, and c are three distinct real numbers. For each nonempty subset of $\{a, b, c\}$, we can compute its average. For example,

$$\text{average}\{a\} = a,$$

$$\text{average}\{a, b\} = \frac{a+b}{2}.$$

If the averages, listed in increasing order, of the 7 nonempty subsets of $\{a, b, c\}$ are

$$2, 6, 10, 12, 13, 17, 24,$$

then what is $\dfrac{a+b+c}{3}$?

(A) 6 (B) 10 (C) 12 (D) 13 (E) 17

Problem 3. If $5^x \cdot 7^y = 4$, then y, as a function of x, is

(A) linear (B) polynomial of degree ≥ 2 (C) exponential
(D) logarithmic (E) y is not a function of x

Problem 4. Tank A is a vertical cylinder with radius 2 ft; tank B is a vertical cylinder with radius 3 ft. Water is drained from tank A into tank B. If the depth of the water in tank A is decreasing at a constant rate of 1 ft/hr, at what rate is the depth of the water in tank B increasing?

(A) 1 ft/hr (B) $\dfrac{2}{3}$ ft/hr (C) $\dfrac{3}{2}$ ft/hr (D) $\dfrac{4}{9}$ ft/hr (E) $\dfrac{9}{4}$ ft/hr

Problem 5. If you cut a finite interval randomly into two pieces, what is the probability that one of them is at least three times as long as the other?

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{2}{3}$ (E) $\dfrac{3}{4}$

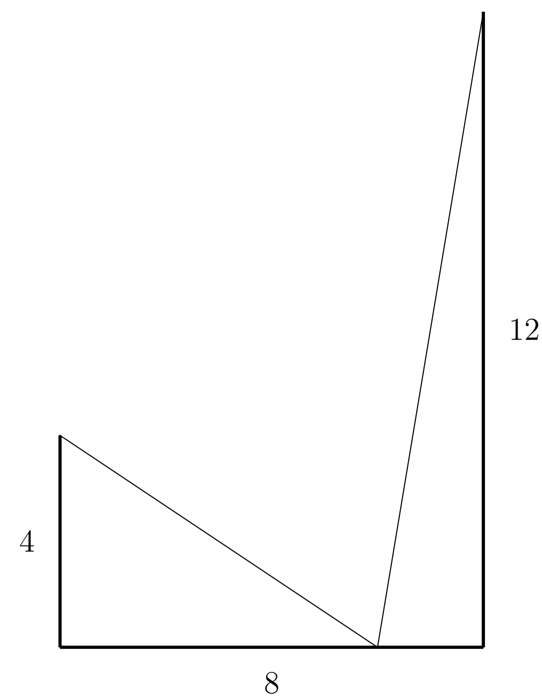
Problem 6. What is the smallest n such that a group of n people has probability $\geq \frac{1}{2}$ of having two people born in the same month, assuming that each of the 12 possible birth months is equally likely?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 7. If $f(x) = e^{-x}$, what is the range of $f(f(f(x)))$?

(A) $(1, e)$ (B) $(1/e, 1)$ (C) $(0, 1/e)$ (D) $(0, 1)$ (E) $(0, e)$

Problem 8. You are connecting a wire between the top of a height 4 ft pole and the top of a height 12 ft pole. The wire also has to be connected to the ground between the poles. If there is 8ft between the poles, what is the minimum length of the wire?

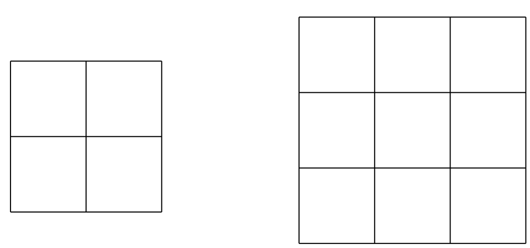


(A) $8\sqrt{5}$ (B) 18 (C) $13\sqrt{2}$ (D) 20 (E) 21

Problem 9. Suppose each positive integer is assigned exactly one of the colors red, black or silver, in such a way that for every $z \geq 3$ there are distinct positive integers x and y so that $x + y = z$, and x, y, and z are all different colors; i.e. $x + y = z$ has a "rainbow solution." If 2019 is red, what color(s) can 1 be?

(A) red (B) black (C) black or silver (D) red, black, or silver
(E) there is no such coloring

Note. For $n > 1$, a geometric figure is called an *n-reptile* if it can be decomposed into n figures which are congruent to each other and similar to the original figure. For example, a square is a 4-reptile, and also a 9-reptile.



You also saw in the ciphering round that a $1 \times \sqrt{2}$ rectangle is a 2-reptile.

Problem 10. What is the smallest $n > 1$ for which a 30°-60°-90° triangle is an n-reptile?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 11. For how many n, $1 < n < 50$, is an equilateral triangle an n-reptile?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 12. What is the smallest positive integer n such that $p(n) = n^2 + 20n + 19$ is divisible by 2019?

(A) 654 (B) 672 (C) 1327 (D) 2000 (E) 2019

Problem 13. For how many positive integers n is the decimal expansion of $\frac{1}{n}$ purely periodic with period 3 (and no smaller period!) ?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 14. If $a = 1 + z$ and $b = 1 - \frac{1}{z}$ and $a + b = 7$, what is $a^3 + b^3$?

(A) 238 (B) 288 (C) 321 (D) 325 (E) 343

Note. The next two problems are about the area of a union of disks. Recall that a *disk* consists of a circle together with the points inside the circle.

Problem 15. Let L be the line segment in $\mathbb{R}^2$ from $(0, 0)$ to $(2, 0)$. At each point $(x, 0)$ of L draw a disk of radius x centered at $(x, 0)$. What is the area of the union of these disks?

(A) $2 + \dfrac{\pi}{2}$ (B) $2 + \pi$ (C) $2 + 2\pi$ (D) 2π (E) 4π

Problem 16. Again let L be the line segment in $\mathbb{R}^2$ from $(0, 0)$ to $(2, 0)$. At each point $(x, 0)$ of L, draw a disk of radius $\frac{x}{2}$ centered at $(x, 0)$. What is the area of the union of these disks?

(A) $\dfrac{\sqrt{3}}{2} + \pi$ (B) $\sqrt{3} + \pi$ (C) $\dfrac{\sqrt{3}}{2} + \dfrac{2}{3}\pi$ (D) $\sqrt{3} + \dfrac{2}{3}\pi$ (E) 2π

Problem 17. The terms in a sequence of positive integers add up to 2019. If P is the largest possible product of these terms, what is the number of different primes dividing P?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 18. Recall that a polygon is said to be *convex* if all of its interior angles measure strictly less than $180°$. A *lattice point* in $\mathbb{R}^2$ is a point with coordinates (x, y), with x and y both integers. Now consider the following statement:

> Every convex n-gon in $\mathbb{R}^2$ having all its vertices at lattice points contains a lattice point in its interior.

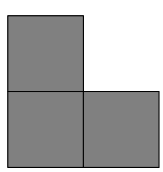
The least positive integer $n \geq 3$ for which this statement holds is

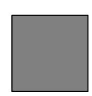
(A) 4 (B) 5 (C) 6 (D) 7 (E) there is no such n

Problem 19. For how many positive integers n is $(3.5)^n + (12.5)^n$ an integer?

(A) 1 (B) 2 (C) 3 (D) 4 (E) infinitely many

Problem 20. Suppose you want to cover an 8×8 checkerboard with 21 L-shaped tiles and a single 1×1 square tile.





How many different locations are there on the board where the square tile could be located in a successful tiling? You are allowed to rotate the L-shaped tile.

(A) 0 (i.e., no such tiling is possible) (B) 4 (C) 8 (D) 22 (E) 64

Problem 21. Suppose you want to cover an 8×8 checkerboard with 21 straight tiles of length 3 and a single 1×1 square tile.

How many different locations are there on the board where the square tile could be located in a successful tiling? You are allowed to rotate the straight tile.

(A) 0 (i.e., no such tiling is possible) (B) 4 (C) 8 (D) 22 (E) 64

Problem 22. For each integer $n > 1$, let $f(n)$ denote the number of *factorizations* of n, meaning the number of ways of writing n as an ordered product of integers larger than 1. For example, 12 has 8 factorizations:

$$12, \quad 2 \cdot 6, \quad 6 \cdot 2, \quad 3 \cdot 4, \quad 4 \cdot 3, \quad 2 \cdot 2 \cdot 3, \quad 2 \cdot 3 \cdot 2, \quad 3 \cdot 2 \cdot 2.$$

Let $f_{\text{odd}}(n)$ count the number of factorizations with an odd number of parts and $f_{\text{even}}(n)$ count the number of factorizations with an even number of parts. What is the largest value of

$$|f_{\text{odd}}(n) - f_{\text{even}}(n)|$$

for $1 < n \leq 2019$?

(A) 0 (B) 1 (C) 3 (D) 39 (E) 40

Problem 23. Recall: If X and Y are sets, we say X is a *subset* of Y if every element of X is also an element of Y; we denote this by $X \subseteq Y$. The empty set $\emptyset$ is a subset of every set. We say X is a *proper subset* of Y if X is a subset of Y and $X \neq Y$; in this case, we write $X \subsetneq Y$.

How many triples of sets A, B, C are there that satisfy

$$\emptyset \subseteq A \subsetneq B \subsetneq C \subseteq \{1, 2, 3, 4, 5\} \quad ?$$

(A) 504 (B) 570 (C) 729 (D) 768 (E) 776

Problem 24. How many ordered triples of integers (x, y, z) are there satisfying the simultaneous conditions

$$x + y + z = 3 \qquad \text{and} \qquad x^3 + y^3 + z^3 = 3 \quad ?$$

The integers x, y, z do not have to be distinct, and are allowed to be positive, negative, or zero.

(A) 1 (B) 4 (C) 7 (D) 10 (E) infinitely many

Problem 25. Suppose $p(x)$ is a degree 6 polynomial which is tangent to $y = 2x + 3$ at $x = 1$, $x = 2$, and $x = 5$. What values can $p(0)$ have?

(A) $p(0) = 3$ (B) $p(0) = 100$ (C) $p(0)$ can be any real number except 3 (D) $p(0)$ can be any real number except 100 (E) there is no such polynomial