Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 20, 2018

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. If $a > 2$ and $(a-2)^a + (a-1)^a + (a+1)^a + (a+2)^a = 2018$, what is a?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 2. A certain recipe calls for butter, eggs, sugar, vanilla, and flour. After the other ingredients are well mixed, the recipe says: "Add $\frac{1}{3}$ of the flour and mix well. Then add $\frac{1}{2}$ the remaining flour and mix well. Then add the rest of the flour, mix well, and bake."

If "the rest of the flour" is 1 cup, how much flour is in the recipe?

(A) $1\frac{1}{2}$ cups (B) $1\frac{2}{3}$ cups (C) $1\frac{5}{6}$ cups (D) 2 cups (E) 3 cups

Problem 3. Each of the following rows contains two functions. For which row(s) are the graphs of the two functions identical?

I. $y = \log\left((x+5)(x^2-16)\right), \quad y = \log(x+5) + \log\left(x^2-16\right)$

II. $y = \log\left((x+5)(x^2-16)\right), \quad y = \log(x+5) + \log(x-4) + \log(x+4)$

III. $y = \log\left((x+3)(x^2-16)\right), \quad y = \log(x+3) + \log\left(x^2-16\right)$

IV. $y = \log\left((x+3)(x^2-16)\right), \quad y = \log(x+3) + \log(x-4) + \log(x+4)$

(A) **I** (B) **II** (C) **III** (D) **IV** (E) The graphs are identical in each row.

Problem 4. 100 students took a test, and their average score was 75 (out of a possible 100). If n is the number of students who scored ≥ 90 what is the largest n can be?

(A) 50 (B) 75 (C) 80 (D) 83 (E) 84

Problem 5. Suppose $x_1, x_2, \ldots, x_n$ is a finite data set with

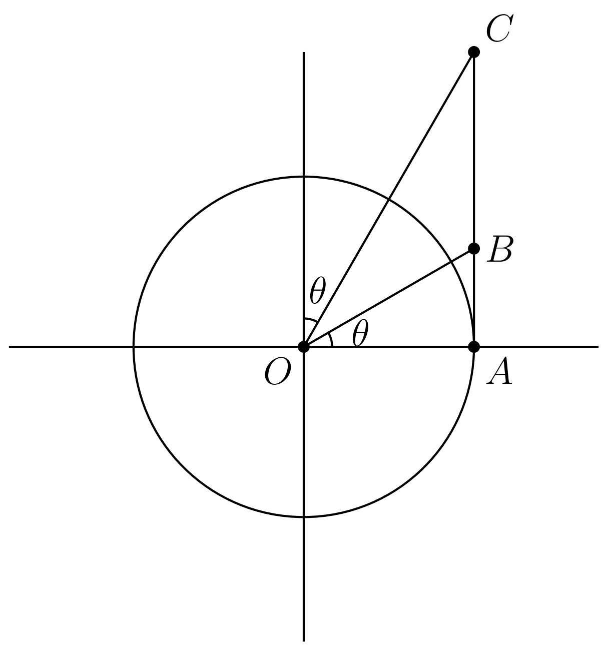
$$\text{minimum} < \text{mean} < \text{median} < \text{mode} < \text{maximum}.$$

What is the smallest n can be?

We use the following definitions: When the data are listed in increasing order, the median is the middle number in that list, or the average of the two middle numbers. The mode is the unique data point that occurs most frequently.

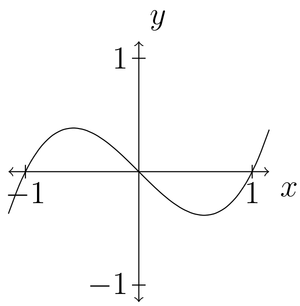
(A) 4 (B) 5 (C) 6 (D) 7 (E) there is no such data set

Problem 6. In the diagram shown $0 < \theta < \dfrac{\pi}{4}$ and the line AC is tangent to the circle at A. Express $\dfrac{OC}{OB}$ as a function of θ.



(A) $\dfrac{\sec(\theta)}{\sqrt{2}}$ (B) $\dfrac{\csc(\theta)}{\sqrt{2}}$ (C) $\tan(\theta)$ (D) $\cot(\theta)$ (E) $\dfrac{\pi}{4\theta}$

Problem 7. Teacher: Here is the graph of $f(x)$.



Teacher: What does the graph of $f\left(x^2\right)$ look like?
Me:

(A) Top left (B) Top right (C) Bottom left (D) Bottom right

(There are only 4 answer choices for this problem.)

Problem 8. Suppose x_n is a sequence of integers which satisfy the usual Fibonacci recurrence:

$$x_{n+1} = x_n + x_{n-1}.$$

What is the smallest possible value of x_1 if x_1 is positive and $x_1 = x_{10}$?

(A) 7 (B) 13 (C) 14 (D) 17 (E) 34

Problem 9. A chocolate bar is a rectangle made up of individual square pieces. You can break a bar into two smaller bars by separating along any row or column that joins pieces together. For this problem, a break can *only affect one bar at a time*; for example you may not stack different bars on top of each other to perform simultaneous breaks.

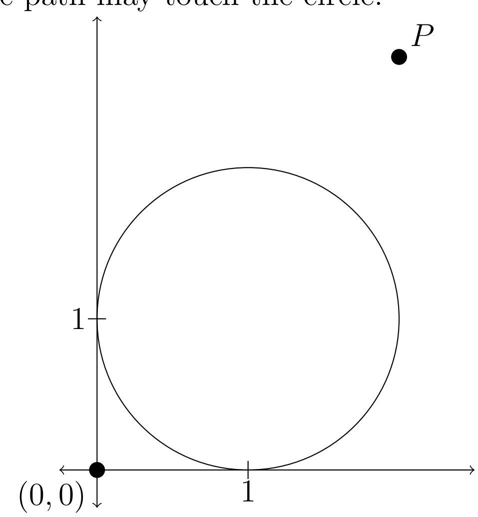
If we start with a 3×4 rectangle, what is the minimum number of breaks required to separate all 12 individual squares?

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Problem 10. For this problem, we want to break a chocolate bar into individual pieces, but we *can* stack different bars on top of each other to break multiple bars simultaneously. With that change to the rules, what is the minimum number of breaks needed to completely separate a 3×4 bar?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 11. The circle shown has radius 1 and center $(1, 1)$. What is the length of the shortest path from $(0, 0)$ to $P = (2, 1 + \sqrt{3})$ that does not go inside the circle. The path may touch the circle.



(A) $1 + \frac{\pi}{2} + \sqrt{3}$ (B) $1 + \frac{\pi}{3} + \sqrt{3}$ (C) $1 + \frac{\pi}{4} + \sqrt{3}$
(D) $1 + \frac{\pi}{6} + \sqrt{3}$ (E) $1 + \frac{\pi}{2} + \sqrt{5 - 2\sqrt{3}}$

Problem 12. How many ordered pairs (x, y) of distinct positive integers satisfy

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2018}?$$

(A) 0 (B) 4 (C) 8 (D) 16 (E) 20

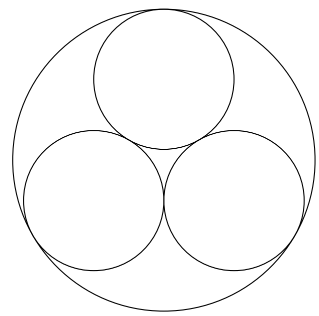
Problem 13. Suppose you're going to choose an integer from 1 to 100, inclusive, and that for each $k = 1, 2, \ldots, 100$, you are k times as likely to choose the number k as you are to choose the number 1. What is the expected value of your choice?

(A) 50 (B) $66.\overline{6}$ (C) 67 (D) 75 (E) 80

Problem 14. Call a composite number *obviously composite* if it is divisible by 2, 3, or 5. How many composite numbers are there in the interval $[2, 1000]$ that are not obviously composite? You may find helpful that the interval $[2, 1000]$ contains 168 prime numbers.

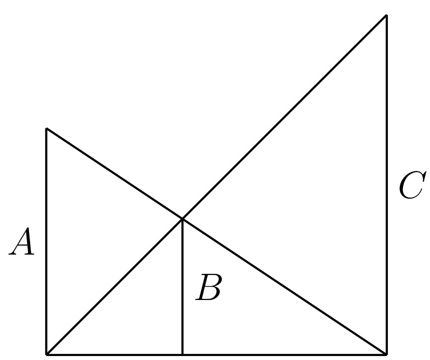
(A) 97 (B) 100 (C) 222 (D) 266 (E) 267

Problem 15. A circle of radius 1 has three congruent mutually tangent circles inscribed in it as shown. What is the radius r of the inscribed circles?



(A) $\frac{1}{3}$ (B) $2-\sqrt{3}$ (C) $\sqrt{2}-1$ (D) $2\sqrt{3}-3$ (E) $3-2\sqrt{2}$

Problem 16. If A, B, and C are the side lengths shown in the diagram, which of the following is true?

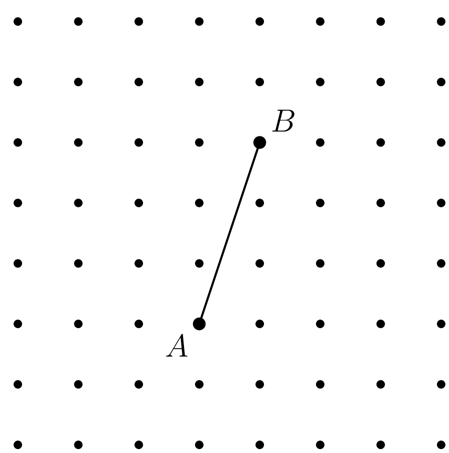


(A) $A+C=4B$ (B) $AC=4B^2$ (C) $A^2+C^2=4B^2$
(D) $\frac{1}{A}+\frac{1}{C}=\frac{1}{B}$ (E) $\frac{AB}{2}+\frac{BC}{2}=AC$

Problem 17. Call a prime *deletable* if it remains prime upon deletion of any proper subset of its (base 10) digits. How many deletable primes exist? *Note: A proper subset means a subset that is not the entire set.*

(A) 7 (B) 8 (C) 9 (D) 10 (E) 12

Problem 18. In the 8 by 8 grid below two points A and B are marked. If point C is one of the other points in the grid, how many choices of C make the triangle ABC acute (i.e. every angle less than 90 degrees)?



(A) 8 (B) 14 (C) 18 (D) 20 (E) 30

Problem 19. What is the largest prime p for which p^2 divides the binomial coefficient $\binom{100}{50}$?

(A) 3 (B) 7 (C) 19 (D) 31 (E) 47

Problem 20. We define a *magic square* to be a collection of nine entries in a 3×3 grid such that the three numbers in each row, column, and diagonal add up to be the same value. If you have a magic square and you know the first two entries in the middle row are 3 and 21 as shown, what is the third entry in that row?

3	21	?

(A) 3 (B) 8 (C) 23 (D) 39 (E) there is more than one possibility

Problem 21. A polynomial $f(x)$ of degree 4, with real number coefficients, has the property that $f(n)$ is an integer whenever n is an integer. Write

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

If $0 \le a_3 \le 1$, then how many possible values are there for a_3?

(A) 1 (B) 2 (C) 6 (D) 7 (E) 13

Problem 22. If $x + \dfrac{1}{x} = 3$, what is $x^{12} + \dfrac{1}{x^{12}}$?

(A) 103682 (B) 103729 (C) 103822 (D) 103823 (E) 104974

Problem 23. The UGA MathClub is proud to announce it has its own cryptarithmetic puzzle:

$$\begin{array}{r} U\,G\,A \\ +\,H\,S\,M\,T \\ \hline M\,A\,T\,H \end{array}$$

A cryptarithmetic puzzle is an arithmetic problem, as above, where each letter represents a single digit (0 - 9). Each occurrence of the same letter must represent the same digit; different letters represent different digits. Also, there are no leading 0's.

If, in the puzzle above, $A = 7$, what is M?

(A) 1 or 2 (B) 3 or 4 (C) 5 or 6 (D) 7 or 8 (E) 9 or 0

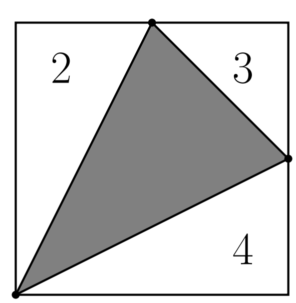
Problem 24. Suppose

$$\sqrt[3]{\frac{a}{b}} = \sqrt[3]{3} + \left(3\sqrt[3]{2} - 3\right)^{\frac{2}{3}},$$

where $\dfrac{a}{b}$ is a rational number in lowest terms. What is $a + b$?

(A) 13 (B) 29 (C) 31 (D) 42 (E) 52

Problem 25. In the rectangle shown at right the three border triangles have areas 2, 3, and 4 as shown (though the figure is certainly not drawn to scale). What is the area of the shaded central triangle?



(A) 5 (B) 6 (C) 7 (D) 8 (E) 9