Topology Qual, August 2024

Complete 8 out of 10 problems (include at least one of the first two) and clearly mark which two problems you do not want us to grade by crossing them out. If you do not cross anything out we will grade the first 8 problems regradles of what you do or don't do on 9 and 10. Each problem is worth 10 points for a maximum possible 80 pts.

Justify all the calculations and state the theorems you use in your answers. You can use any standard algebra results. You may also assume the fundamental group and homology groups of a point, the 2-torus, and spheres of all dimensions. All other groups used should be computed/explained.

(1) (a) Give an example of a continuous bijection that is not a homeomorphism.
(b) Prove that a continuous bijection $f : X \to Y$ from a compact space X to a Hausdorff space Y is a homeomorphism.

(2) (a) Show that a path connected space is connected.
(b) Show that a connected and locally path connected space is path connected.

(3) Let X, Y, Z denote closed (compact, without boundary) surfaces.
(a) Prove that if there exists a covering map $f : X \to Y$ of degree n, then $\chi(X) = n\chi(Y)$.
(b) Prove that if both $f : X \to Y$ and $g : Y \to Z$ are covering maps, then $g \circ f : X \to Z$ is also a covering map. Furthermore, prove that the degree of $g \circ f$ is the product of the degrees of f and g.

(4) Describe all surfaces, orientable or non-orientable, with boundary or closed, that have $\chi(X) = -2$.

(5) Let $X = S^1 \times S^1$ and $Y = S^1 \vee S^1 \vee S^2$. Show that X and Y have isomorphic homology groups but are not homotopy equivalent.

(6) Suppose that a continuous map $f : S^3 \times S^3 \to \mathbb{R}P^3$ is not surjective. Prove that f is homotopic to a constant function.

(7) Let X_n be the topological space obtained by identifying $n > 1$ points on S^2 to a single point. Describe a CW decomposition of X_n and use it to calculate the fundamental group and homology groups of X_n.

(8) Consider the connect sum $X = RP^2 \# RP^2$ as union of two homeomorphic pieces glued across the boundary circle. Explain what the pieces are and apply the Mayer-Vietoris long exact sequence to this decomposition to calculate the homology groups of X and Van Kampen's theorem to calculate $\pi_1(X)$.

(9) Recall that RP^n is the quotient of S^n by the antipodal map. Prove that for $n \geq 2$ there cannot exist a pair of open sets $U, V \subset RP^n$ such that
(a) $U \cup V = RP^n$
and
(b) U and V are both contractible.

(10) Let S be an oriented surface and let $f : S \to S$ be a continuous map homotopic to the identity that has no fixed points. Find all the possible values for the genus of S. Prove your answer is true.