1785

Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 24, 2015

WITH SOLUTIONS

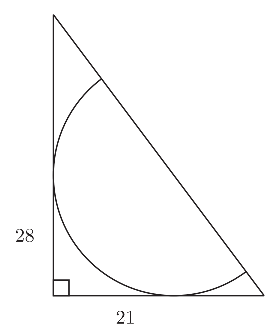
1 Easy Problems

Problem 1. How many prime numbers can be written both as a sum and as a difference of two prime numbers?

(A) 0 (B)$^\heartsuit$ 1 (C) 2 (D) 4 (E) infinitely many

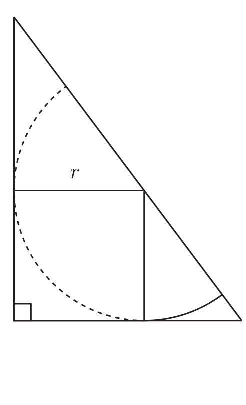
Solution. If p is such a prime, then p is odd, since 2 is not a sum of primes. So p can be written as the sum of an even prime and an odd prime, say $p = p_1 + 2$. Similarly, $p = p_2 - 2$ for some prime p_2. Thus, $p-2, p$, and $p+2$ are all prime. This happens only when $p = 5$ and $\{p-2, p, p+2\} = \{3, 5, 7\}$.

Problem 2. A semicircle is drawn in a right triangle, tangent to both legs, and with a diameter on the hypotenuse. If the legs have length 21 and 28, what is the radius of the semicircle?



(A) 10 (B) 11 (C)$^\heartsuit$ 12 (D) 13 (E) 14

Solution. Draw radii from the center of the circle to the points of tangency, and notice that these are part of an r by r square. Now apply the law of similar triangles:



$$\frac{28}{21} = \frac{28-r}{r}.$$

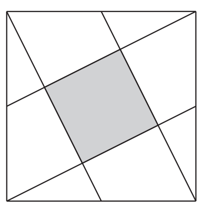
Solving for r gives $r = 12$.

Problem 3. Priya drove her Prius for 30 minutes at 60 miles/hour (mph), and her gas mileage was 45 miles/gallon (mpg). She then slowed down to 50 mph for the next hour, and her fuel consumption for that hour improved to 50 mpg. What was Priya's average fuel consumption, in mpg, for the entire trip?

(A) 46 (B) $47\frac{3}{11}$ (C) $47\frac{1}{2}$ (D)$^\heartsuit$ 48 (E) $48\frac{1}{3}$

Solution. Total fuel consumed: $\frac{\frac{1}{2}\text{ hr}\cdot 60\text{ mph}}{45\text{ mpg}} + \frac{1\text{ hr}\cdot 50\text{ mph}}{50\text{ mpg}} = \frac{5}{3}$ gal.
Total miles traveled: $\frac{1}{2}$ hr $\cdot$ 60 mph $+$ 1 hr $\cdot$ 50 mph $=$ 80 miles.
Hence, Priya's average fuel consumption was $\frac{80\text{ mi}}{\frac{5}{3}\text{ gal}} = 48$ mpg.

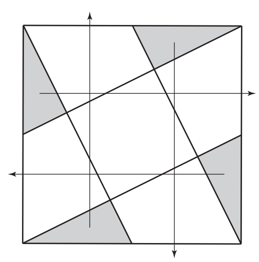
Problem 4. In the diagram shown, each vertex of the larger square is connected to the midpoint of a side.

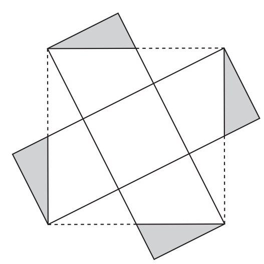


If the larger square has area 1, what is the area of the shaded square?

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C)$^\heartsuit$ $\frac{1}{5}$ (D) $\frac{1}{6}$ (E) none of these

Solution. Consider the 4 shaded regions in the left diagram. Move them as indicated by the arrows.





The resulting 5-square cross has the same area as the original square (i.e., area 1), so each of the smaller squares has area $\frac{1}{5}$.

Problem 5. For how many natural numbers $n \leq 25$ is $(n-1)!$ not divisible by n?

(A) 9 (B)$^\heartsuit$ 10 (C) 12 (D) 13 (E) 14

Solution. Clearly, n does not divide $(n-1)!$ if n is prime. If n factors as a product of two distinct factors (e.g., $2 \cdot 4$), both of those factors appear in $(n-1)!$, and so n does divide $(n-1)!$. This leaves only the squares of primes, $2^2, 3^2, 5^2$ to consider. Of these, 4 does not divide 3!, but the others work. So the only such n are $2, 3, 4, 5, 7, 11, 13, 17, 19, 23$ — 10 of them.

Problem 6. If r_1 and r_2 are the (real or complex) solutions to $ax^2+bx+c=0$, what is $r_1^2+r_2^2$?

(A)$^\heartsuit$ $\frac{b^2-2ac}{a^2}$ (B) $\frac{b^2+2ac}{a^2}$ (C) $\frac{-b^2+2ac}{a^2}$ (D) $\frac{b^2-4ac}{a^2}$ (E) $\frac{b^2-4ac}{2a}$

Solution. Writing $ax^2+bx+c = a(x-r_1)(x-r_2)$ and comparing coefficients, $r_1r_2 = c/a$ and $r_1+r_2 = -b/a$. Hence,

$$r_1^2+r_2^2 = (r_1+r_2)^2 - 2r_1r_2 = (-b/a)^2 - 2c/a = \frac{b^2-2ac}{a^2}.$$

Problem 7. You have to climb a staircase with infinitely many steps according to the following pattern: 11 steps up, 8 steps down, repeat. If you start at the first step, how many times will you pass the 2015th step of the staircase?

(A)$^\heartsuit$ 7 (B) 6 (C) 5 (D) 4 (E) 3

Solution. By convention, when we say that you are *at the nth step*, we mean that you are standing on the nth step after coming back 8 steps.

- When you are at step 2005, you did not touch step 2015 yet.
- When you are at step 2008, you passed step 2015 twice (on your way up and on your way down).
- When you are at step 2011, you passed step 2015 2 more times (on your way up and on your way down).
- When you are at step 2014, you passed step 2015 2 more times (on your way up and on your way down).
- When you are at step 2017, you passed step 2015 another 1 time (on your way up).

In conclusion, you passed the 2015th step 7 times.

Problem 8. A confused student wanted to solve an equation of the form

$$(x-3)(b-x) = 6$$

for the variable x. He tried to find the two solutions by instead solving the two equations

$$\begin{aligned} x-3 &= 6 \\ b-x &= 6. \end{aligned}$$

Surprisingly, he got both solutions correct! What is b?

(A) 0 (B) 3 (C) 6 (D) 9 (E)$^{\heartsuit}$ 10

Solution. From $x - 3 = 6$, we know that $x = 9$ is a solution. Substitute into the original equation to get

$$(9 - 3)(b - 9) = 6.$$

So $b = 10$.

Remark. More generally, the equation $(x - a)(b - x) = c$ has the same solutions as

$$x - a = c$$
$$b - x = c$$

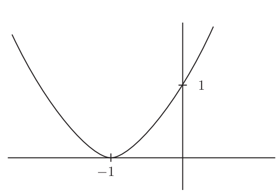
if $c = 0$ or if $c = b - a - 1$. Surprisingly, almost every quadratic equation can be rearranged (using correct algebra) to the form $(x - a)(b - x) = c$ with $c \neq 0$ in such a way that the roots of the quadratic are the same as the solutions to $x - a = c$ and $b - x = c$. The exceptions: quadratics whose roots differ by 1.

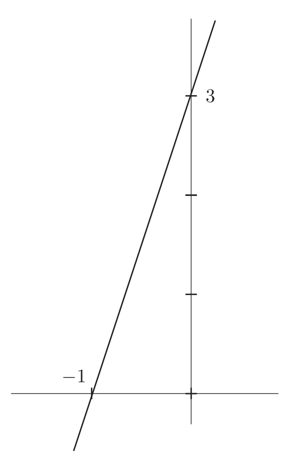
Problem 9. How many polynomials with nonnegative integer coefficients have $p(10) = 200$?

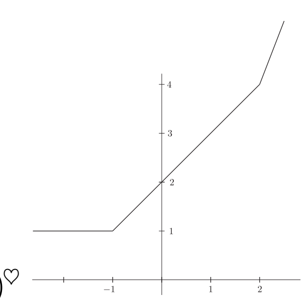
(A) 22 (B) 31 (C) 32 (D)$^{\heartsuit}$ 33 (E) infinitely many

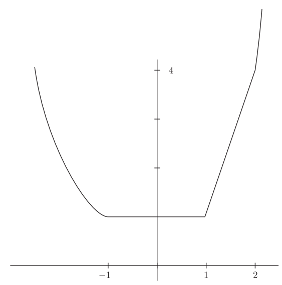
Solution. Organize by degree. There is only one degree 0 (constant) polynomial with $p(10) = 200$, namely $p(x) = 200$. If $p(x)$ is linear, then $p(x) = ax + b$, so $p(10) = 10a + b$. For each $a = 1, 2, \ldots, 20$, there is a unique nonnegative integer value of b with $10a + b = 200$. Hence, there are 20 linear polynomials. Among the quadratic polynomials, $p(x) = 2x^2$ has $p(10) = 200$, along with any $p(x) = x^2 + ax + b$ satisfying $10a + b = 100$. Counting as in the constant and linear cases, there are 11 such polynomials $x^2 + ax + b$. So in total, there are $1 + 20 + 1 + 11 = 33$ polynomials.

Problem 10. In *tropical arithmetic*, $a + b$ means the maximum of a and b, while $a \cdot b$ means the sum of a and b. So, for example, $2 + 3 = 3$ and $5^2 = 10$. What is the graph of the tropical polynomial $x^2 + 2x + 1$?

(A) 

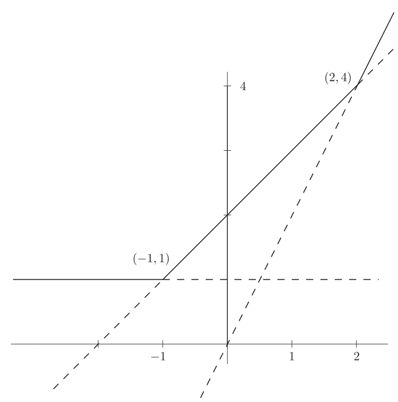
(B) 

(C)$^{\heartsuit}$ 

(D) 

(E) none of the above

Solution. $x^2 + 2x + 1 = \max\{2x, x + 2, 1\}$. Graph the 3 lines $y = 2x, y = x + 2$, and $y = 1$ together, then take the highest point above each x-value:



Remark. *Tropical geometry* is an area of mathematics which studies the geometric and combinatorial properties of tropical polynomials. A *tropical polynomial* is nothing more than a usual polynomial where addition and multiplication are as described above. Tropical geometry turns out to have important interactions with biology.

2 Medium Problems

Problem 11. Suppose `ab` is a 2 digit number with the property that the 6 digit number `1234ab` is divisible by 9 and `ab1234` is divisible by 11. What is $a^2 - b^2$?

(A)$^\heartsuit$ 16 (B) 34 (C) -11 (D) -72 (E) -153

Solution. `1234ab` is divisible by 9 iff $1 + 2 + 3 + 4 + a + b = 10 + a + b$ is divisible by 9, so $10 + a + b = 18$ or 27, i.e., $a + b = 8$ or 17. Similarly, `ab1234` is divisible by 11 iff $a - b + 1 - 2 + 3 - 4 = a - b - 2$ is divisible by 11, so $a - b - 2 = -11$ or 0, i.e., $a - b = -9$ or 2. Notice that $a + b = 17$ implies that $\{a, b\} = \{8, 9\}$, so $a - b$ is neither -9 nor 2. Similarly, $a - b = -9$ implies that $a = 0$ and $b = 9$, so that $a + b$ is neither 8 nor 17. So $a + b = 8$ and $a - b = 2$, and $a^2 - b^2 = 16$. In fact, `ab=53`.

Problem 12. In writing this test, we create 25 problems and then rank them. 10 should be "easy", 10 "medium", and 5 "hard". When doing this, we wondered how many ways there are to put the 25 problems into these 3 groups.

Luca claimed that it could be done in $\binom{25}{15}\binom{15}{10}$ ways.

Mo claimed that it could be done in $\binom{25}{10}\binom{15}{10}$ ways.

Paul claimed that it could be done in $\binom{25}{5}\binom{20}{10}$ ways.

Ted claimed that it could be done in $\binom{25}{10}\binom{15}{5}$ ways.

Here $\binom{n}{k}$ is the binomial coefficient, representing the number of ways of choosing k elements from an n-element set.

How many of these claims are correct?

(A)$^\heartsuit$ 4 (B) 3 (C) 2 (D) 1 (E) 0

Solution. They are all correct.

Luca chose 15 problems that were medium or hard, then chose 5 (of those 15) as hard.

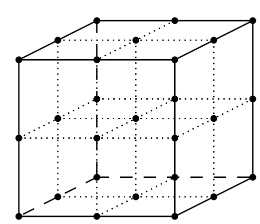
Mo chose 10 easy problems, then chose 10 of the remaining 15 as medium.

Paul chose the 5 hard problems, then 10 of the remaining 20.

Ted chose 10 easy problems, then chose 5 of the remaining 15 as hard.

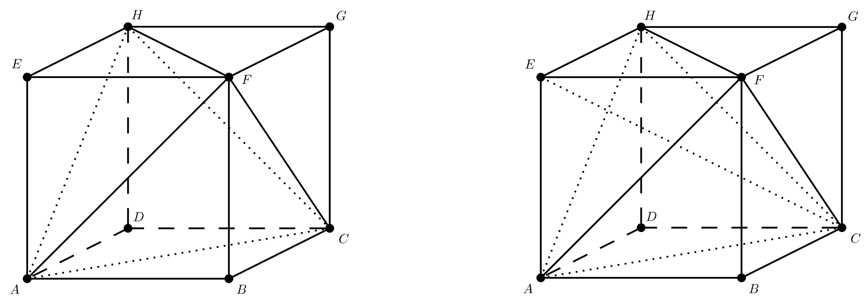
It is also easy to see that Luca, Mo, and Ted have the same answer by the symmetry of the binomial coefficients: $\binom{n}{k} = \binom{n}{n-k}$.

Problem 13. Suppose you want to cover the cube in the figure with tetrahedra having the black dots as possible vertices. What is the maximum number of tetrahedra you can use assuming that the intersection of two tetrahedra is either empty, a vertex, an edge, or a face of both of them?



(A) 8 (B) 24 (C) 32 (D)$^\heartsuit$ 48 (E) 64

Solution. To start with, one can have a reasonable guess of what the answer is. The big cube in the figure is made of 8 smaller cubes. Now, what is the maximum number of tetrahedra we can use to cover one of these smaller cubes? A first attempt can be the subdivision into 5 tetrahedra in the left figure below. But from this subdivision we can get 6 tetrahedra by adding the diagonal CE, as in the right-hand figure.



The 6 tetrahedra in the picture are $ABCF, CFGH, ACDH, CEFH, ACEF$ and $ACEH$. In this way we covered our big cube with 48 tetrahedra.

Now let us try to understand why this is the maximum possible number. Assume the edge length of the big cube is 2, so that its volume is 8. If we show that the smallest tetrahedron with vertices in the grid has volume $\frac{1}{6}$ we are done.

This can be done by inspection, or it can be proved by using the following fact: the volume of a tetrahedron in a 3-dimensional space is equal to $\frac{1}{6}$ the absolute value of the determinant of the matrix

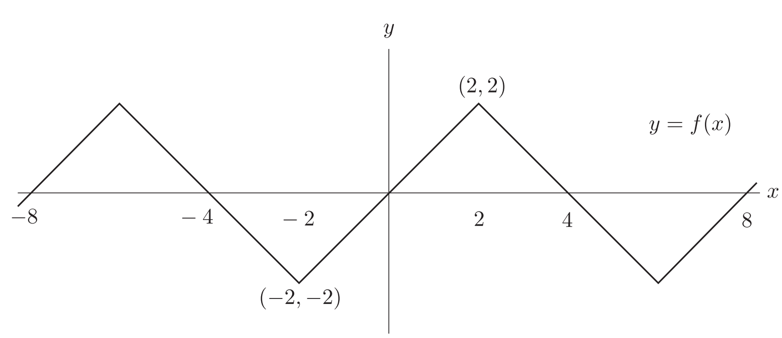
$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

where $(a_1, b_1, c_1), \ldots, (a_4, b_4, c_4)$ are the coordinates of the vertices of the tetrahedron.

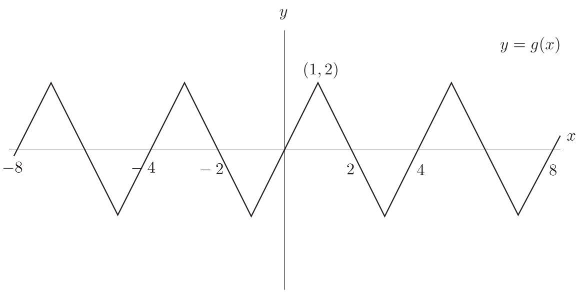
Observe that in this case the number we obtain for the absolute value of the determinant is a positive integer (assume the points of the grid have integers as coordinates). It follows that the volume of a tetrahedron with vertices in the grid is at least $\frac{1}{6}$.

To see that the minimum volume is actually $\frac{1}{6}$, it will be enough to find an example of a tetrahedron with volume $\frac{1}{6}$ and vertices in the grid. However, all the tetrahedra used in the previous decomposition into 48 tetrahedra above have volume $\frac{1}{6}$ as one can easily check.

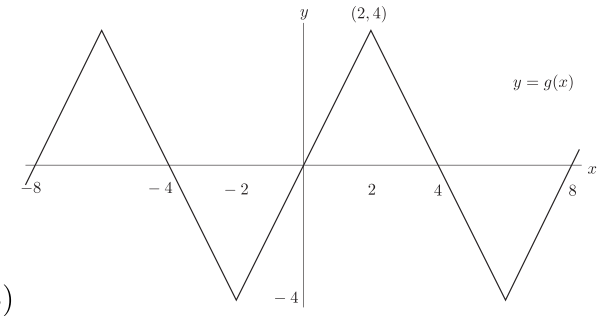
Problem 14. The following is the graph of a certain function $f(x)$:



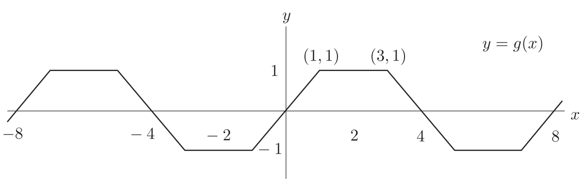
Which of the following is the graph of $g(x) = \dfrac{f(x+1) + f(x-1)}{2}$?



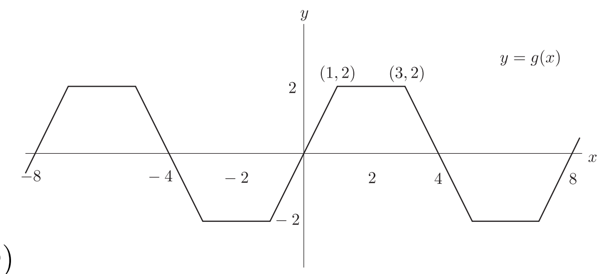
(A)



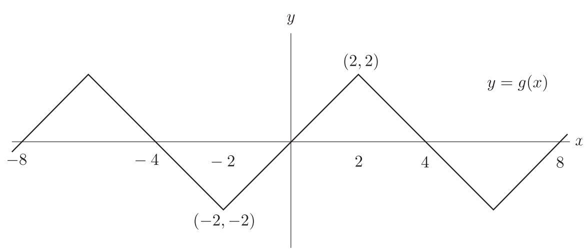
(B)



(C)

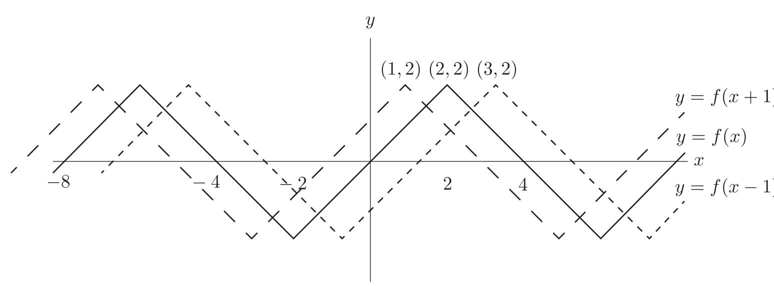


(D)



(E)

Solution. Look at the graphs of $f(x)$, $f(x+1)$, and $f(x-1)$ on the same axes:

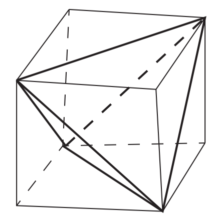


Then $g(x)$ is the average of the two dashed graphs. On the intervals where the two dashed graphs are parallel (e.g., $[-1,1]$), their average is the parallel line halfway between the two. This coincides with $f(x)$. On the intervals on which $f(x+1)$ and $f(x-1)$ are not parallel, the average is constant. The average can't be the same as the maximum of f, eliminating (D).

Remark. D'Alembert showed that the solution to the homogeneous wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ with initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$ is $[f(x+t) - f(x-t)]/2$. This means that a taut string initially shaped like the graph of $f(x)$ will, when released, immediately decompose into a superposition of two waves, each identical to but half the size of $f(x)$, and moving in opposite directions!

Problem 15. A regular tetrahedron of edge length 1 is inscribed in a sphere. What is the radius of the sphere?

(A) $\frac{\sqrt{3}}{2}$ (B)$^\heartsuit$ $\frac{\sqrt{6}}{4}$ (C) $\frac{\sqrt{6}}{2}$ (D) 1 (E) $\sqrt{3}$



Solution. Recall (ciphering problem #6) that a cube with edge length 1 can be inscribed in a sphere of radius $\frac{\sqrt{3}}{2}$, since the diagonal of the cube is a diameter of the sphere.

Now notice that there is a tetrahedron inside the cube; take a diagonal of the top face, and the "other" diagonal of the bottom face. These span a regular tetrahedron with edge length $\sqrt{2}$. This is inscribed in a sphere of radius $\frac{\sqrt{3}}{2}$. So the tetrahedron of edge length 1 can be inscribed in a sphere of radius $\frac{\sqrt{3}/2}{\sqrt{2}} = \frac{\sqrt{6}}{4}$.

Problem 16. Which of the following tests will determine whether a number, written in base 3, is even (divisible by 2)?

I. the rightmost digit is 0 or 2,

II. the sum of the digits is even,

III. the alternating sum of the digits is even.

(A) I only (B) II only (C) III only (D) I and II only
(E)$^\heartsuit$ II and III only

Solution. Write down the base 3 expansion of n, say $n = a_0 + a_1 \cdot 3 + \cdots + a_k 3^k$, where each $a_i \in \{0, 1, 2\}$. Reducing modulo 2, we find

$$n \equiv a_0 + a_1 + \cdots + a_k \pmod{2}.$$

So n is even precisely when the sum of its base 3 digits is even. Since $1 \equiv -1 \pmod{2}$, we also have

$$n \equiv a_0 - a_1 + \cdots + (-1)^k a_k \pmod{2},$$

and so n is even precisely when the alternating sum of its base 3 digits is even. So II and III are valid tests. To see that I is not valid, one only has to note that 4 is even but ends in the digit 1 in base 3.

Problem 17. At almost every point P on the circle, there is a unique line $y = mx + b$ which is tangent to the circle at P. Of course, m and b depend on P. If we plot the points $(m(P), b(P))$ in the (m, b)-plane, what is the shape of the resulting curve?

(A) circle (B) union of lines (C) ellipse but not a circle (D) parabola (E)$^\heartsuit$ hyperbola

Solution. The points of the circle can be parametrized by $(\cos(t), \sin(t))$ for $0 \le t \le 2\pi$. At the point $P(t) = (\cos(t), \sin(t))$, the radius has slope $\frac{\Delta y}{\Delta x} = \frac{\sin(t)}{\cos(t)} = \tan(t)$, so the tangent line has slope $-\cot(t)$. Thus, the equation of the tangent line is

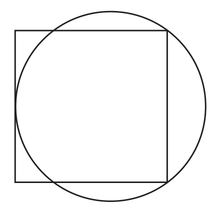
$$y - \sin(t) = -\cot(t)(x - \cos(t)),$$

or (simplifying)

$$y = -\cot(t)x + \csc(t).$$

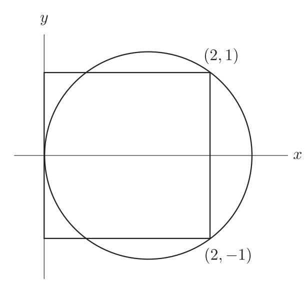
So the points in the (m, b) plane are parametrized by $(-\cot(t), \csc(t))$. The familiar identity $1 + \cot(t)^2 = \csc(t)^2$ shows that these points satisfy $1 + m^2 = b^2$, so lie on the hyperbola $b^2 - m^2 = 1$.

Problem 18. Start with a circle of radius 1, and draw a square so that two adjacent corners of the square lie on the circle, and the opposite side of the square is tangent to the circle. What is the side length of the square?



(A) 2 (B) $\frac{3}{2}$ (C) $\frac{5}{3}$ (D)$^\heartsuit$ $\frac{8}{5}$ (E) $\frac{13}{8}$

Solution. Change the scale so that the square has side length 2 and put the point of tangency at the origin of the (x, y) plane as shown. Then the circle passes through the points $(0, 0)$, $(2, 1)$, and $(2, -1)$. The center of the circle is at $(r, 0)$ on the x-axis. So $r = d((0, 0), (r, 0)) = d((r, 0), (2, 1)) = \sqrt{(r - 2)^2 + 1}$. Solving we get $r = 5/4$. Now scale by a factor of $4/5$ and you'll see a square of side length $\frac{8}{5}$ and a circle of radius 1.



Problem 19. Begin with a solid cylinder of radius r and height h. Intersect with a plane that contains the diameter of the top circle and is tangent to the bottom circle. What is the area of the intersection?

(A) $\pi r h$ (B) $\pi r \sqrt{r^2 + h^2}$ (C) $r\sqrt{r^2 + h^2}$ (D) $\frac{1}{2}\pi r h$ (E)$^\heartsuit$ $\frac{1}{2}\pi r\sqrt{r^2 + h^2}$

Solution. If the cylinder extended farther up, the intersection would be an ellipse with minor axis the diameter. The major axis extends from the point of tangency through the center of the diameter. So we need the area of half an ellipse with semiminor axis r and semimajor axis $\sqrt{r^2 + h^2}$: $A = \frac{1}{2}\pi r\sqrt{r^2 + h^2}$.

Problem 20. Suppose you rotate the graph of $y = x^3$ clockwise through an angle of $45°$ around the origin. The resulting curve is the graph of what polynomial?

(A) $x^3 - x$ (B) $x^3 - 2x$ (C) $\frac{1}{2}x^3 - x$ (D) it is not the graph of a function
(E)$^\heartsuit$ it is the graph of a function but not a polynomial

Solution. If the curve were not the graph of a function, then there would be a line parallel to $y = -x$ intersecting the graph of $y = x^3$ in more than one point. In other words, $x^3 + x = c$ would have more than one solution x for a certain value of c. But $x^3 + x$ is an increasing of function of x, and so there can never be more than one solution to such an equation. So the curve is the graph of a function $f(x)$.

However, $f(x)$ is not a polynomial. Notice that $f(x) < x$ for all $x > 0$: Indeed, if $y = f(x)$ were to ever meet $y = x$, then x^3 would meet the y-axis at the corresponding point. Since $f(x)$ is positive for all $x > \sqrt{2}$ and $f(x) < x$ for all $x > 0$, if $f(x)$ were a polynomial it would necessarily be a linear polynomial. But it is clear that rotating the graph of $y = x^3$ does not give a straight line.

3 Hard problems

Problem 21. Evaluate

$$\sum_{1 \le n \le 100} \left\lfloor \sqrt{100/n} \right\rfloor = \left\lfloor \sqrt{100/1} \right\rfloor + \left\lfloor \sqrt{100/2} \right\rfloor + \cdots + \left\lfloor \sqrt{100/100} \right\rfloor .$$

Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(A)$^\heartsuit$ 153 (B) 167 (C) 180 (D) 199 (E) 200

Solution. Note that the function $\lfloor \sqrt{100/n} \rfloor$ only takes on integer values from 1 to 10. Let's count how many times each value y occurs:
$y = 10$ occurs only when $n = 1$.
y is never 9 or 8.
$y = 7$ when $n = 2$.
y is never 6.
y is 5 when $n = 3$ or $n = 4$.
y is 4 when $n = 5$ or $n = 6$.
y is 3 when $n = 7, 8, 9, 10, 11$.
y is 2 when $n = 12, \ldots, 25$.
y is 1 when $n = 26, \ldots, 100$.
So the sum is

$$10 \cdot 1 + 7 \cdot 1 + 5 \cdot 2 + 4 \cdot 2 + 3 \cdot 5 + 2 \cdot 14 + 1 \cdot 75 = 153.$$

Problem 22. If you list in increasing order all the positive integers that can be written as the sum of distinct nonnegative integer powers of 3, what number is 50th on the the list?

(A) 283 (B)$^\heartsuit$ 327 (C) 337 (D) 356 (E) 364

Solution. We are looking for the 50th number whose base 3 expansion has only zeros and ones. If we list all the numbers whose base 3 expansion omits the digit 2, the expansions look like binary expansions. Since 50 has binary expansion 110010, the 50th number on our list is

$$1 \cdot 3^5 + 1 \cdot 3^4 + 0 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3 + 0 \cdot 1 = 327.$$

Problem 23. Evaluate

$$\sum_{j=0}^{\infty} \frac{(1/3)^{2^j}}{1-(1/3)^{2^{j+1}}} = \frac{1/3}{1-(1/3)^2} + \frac{(1/3)^2}{1-(1/3)^4} + \frac{(1/3)^4}{1-(1/3)^8} + \cdots.$$

(A)$^\heartsuit$ $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) 1 (D) $\frac{3}{2}$ (E) $\frac{4}{3}$

Solution. It is simpler to solve a more general problem: We determine $\sum_{j=0}^{\infty} \frac{x^{2^j}}{1-x^{2^{j+1}}}$ whenever $|x| < 1$. For each nonnegative integer j, we can expand the jth term as a geometric series:

$$\frac{x^{2^j}}{1-x^{2^{j+1}}} = x^{2^j} + x^{2^j+2^{j+1}} + x^{2^j+2\cdot 2^{j+1}} + x^{2^j+3\cdot 2^{j+1}} + \cdots.$$

The exponents here are the numbers of the form $2^j(1+2k)$ for $k = 0, 1, 2, \dots$. Equivalently, they are the positive integers divisible by 2^j but not 2^{j+1}.

For every positive integer n, there is a unique nonnegative integer j for which 2^j divides n but 2^{j+1} does not. So the term x^n appears above for exactly one j. Thus,

$$\sum_{j=0}^{\infty} \frac{x^{2^j}}{1-x^{2^{j+1}}} = x + x^2 + x^3 + \cdots = \frac{x}{1-x}.$$

To finish, we take $x = \frac{1}{3}$, which gives the sum of the infinite series as $\frac{1/3}{1-1/3} = \frac{1}{2}$.

Problem 24. How many odd numbers are there in the 125th row of Pascal's triangle? (Here we number so that the first row is the row $1, 1$.)

(A) 2 (B) 32 (C) 63 (D)$^\heartsuit$ 64 (E) 126

Solution. Since $128 = 2^7$, we know the 128th row of Pascal's triangle is all even aside from the initial and terminal 1's. (For how we know this, see the end of this solution.) We'll represent this schematically by

$$1000\cdots 0001.$$

So the 127th row must look like this:

$$111 \cdots 111 \qquad \text{(row 127)}$$
$$1000 \cdots 0001 \qquad \text{(row 128)}$$

(You should check now that any entry in the 128th row is the sum of the two entries above it, modulo 2.) Similarly, working backwards, we can reconstruct the 125th row:

$$\begin{array}{cr} 11001100 \cdots 00110011 & \text{(row 125)} \\ 101010101 \cdots 101010101 & \text{(row 126)} \\ 1111111111 \cdots 1111111111 & \text{(row 127)} \\ 10000000000 \cdots 0000000001 & \text{(row 128)} \end{array}$$

So the 125th row has 126 entries, and there are two more odd than even. So there are 62 even entries, and there are 64 odd entries.

At the start, we said that the 128th row was all even, except for the outer entries. Why is this true? First look at the second row: 1, 2, 1. This says that $(1+x)^2 = x^2+2x+1$, and so modulo 2, we have $(1+x)^2 = 1+x^2$. So (again mod 2),

$$(1+x)^4 = ((1+x)^2)^2 = (1+x^2)^2 = 1+x^4.$$

Squaring again,

$$(1+x)^8 = ((1+x)^4)^2 = (1+x^4)^2 = 1+x^8.$$

Continuing in this way, $(1+x)^{2^j} = 1+x^{2^j}$ for every j. So the only odd entries in the 2^jth row of Pascal's triangle are the outer 1s.

Problem 25. A group of math students decided to play a game of `BizzBuzz`. Here are the rules:

Players sit in a circle and take turns saying either a number or a word.

The first player must start with 1.

The nth player must say the number n, **except**:

- if n is even, she must say Bizz,
- if n is divisible by 3, she must say Buzz,
- if n is divisible by both 2 and 3, she must say BizzBuzz,
- if n is divisible by 5, she says n.

Each rule overrules the preceding. The game starts like this: 1, Bizz, Buzz, Bizz, 5, BizzBuzz, 7, Bizz, Buzz, 10, 11, BizzBuzz, 13, Bizz, 15. What is the 2015th number that will be said (assuming correct play)?

(A) 4030 (B)$^\heartsuit$ 4319 (C) 4320 (D) 4321 (E) 6045

Solution. This is equivalent to: If we thin out the sequence of positive integers $1, 2, 3, 4 \ldots$ by removing all multiples of 2 and 3, except that we keep all multiples of 5, what is the 2015th term of the remaining sequence? So let us begin by counting how many numbers up to n remain on the list. Call this $R(n)$. By the inclusion-exclusion principle,

$$R(n) = n - \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n}{10} \right\rfloor + \left\lfloor \frac{n}{15} \right\rfloor - \left\lfloor \frac{n}{30} \right\rfloor .$$

Note that $R(30)$ is easy to compute: $R(30) = 14$. And $R(30k) = 14k$ for every positive integer k. Now 2015 is not a multiple of 14, but 2016 is: $2016 = 14 \cdot 144$. So $R(30 \cdot 144) = 2016$. So the 2016th number said is 4320. The preceding number, 4319, is neither even nor a multiple of 3, so it is the 2015th number said.

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Sources. Problem #9 is adapted from a problem in James Tanton's MAA lecture "100 problems about the number 100": `http://www.jamestanton.com/wp-content/uploads/2015/07/MathFest_100-Problems.pptx`. Problem #2 is from I Giochi di Archimede - Gara Triennio, Problem 13, November 17, 2004. Problem #18 was suggested by Cameron Bjorklund and Casey Bowman. Problem #22 was suggested by Neil Lyall.