1785

Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

November 8, 2014

WITH SOLUTIONS

1 Easy Problems

Problem 1. How many ordered triples of prime numbers (x, y, z) are there with $x^y - z = 1$ and and $z \leq 2014$?

(A) 0 (B) 1 (C) 2 (D) 3 (E)$^\heartsuit$ 4

Solution. Rearranging gives $x^y - 1 = z$. Since $x, y > 1$ and $x - 1$ divides $x^y - 1$, the only way $x^y - 1$ can be prime is if $x = 2$. This reduces the problem to counting the number of prime values of y for which $2^y - 1$ is a prime number in $[2, 2014]$. The restriction $2^y - 1 \leq 2014$ forces $y \leq 10$. Checking $y = 2, 3, 5, 7$, we find that $2^y - 1$ is prime for all 4 of these values.

Problem 2. Two points A and B lie on the graph of $y = x^3$. If their x-coordinates differ by 1, what is the least by which their y-coordinates can differ?

(A) 0 (B) $\frac{1}{8}$ (C)$^\heartsuit$ $\frac{1}{4}$ (D) $\frac{1}{2}$ (E) 1

Solution. Call the points (a, a^3) and $(a+1, (a+1)^3)$; we want the smallest value of $(a+1)^3 - a^3 = (a^3 + 3a^2 + 3a + 1) - a^3 = 3a^2 + 3a + 1$. This quadratic assumes its minimum value at the vertex of the corresponding parabola, located at $a = -\frac{3}{2 \cdot 3} = -\frac{1}{2}$. So the points are $(\frac{1}{2}, \frac{1}{8})$ and $(-\frac{1}{2}, -\frac{1}{8})$, and the difference in their y-coordinates is $\frac{1}{4}$.

Problem 3. Which real number is the value of the following infinite product?

$$(1 + \frac{1}{2}) \times (1 + \frac{1}{2^2}) \times (1 + \frac{1}{2^4}) \times (1 + \frac{1}{2^8}) \times \cdots$$

(A) 1 (B) $\frac{1+\sqrt{5}}{2}$ (C)$^\heartsuit$ 2 (D) e (E) diverges to infinity

Solution. Expanding the infinite product, we obtain the sum of all expressions of the form

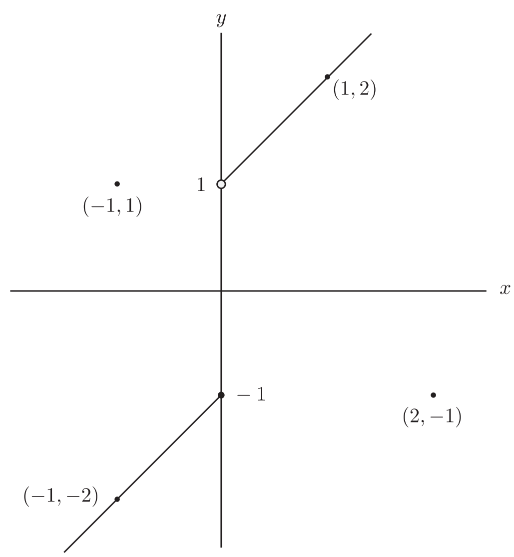
$$\frac{1}{2^{\epsilon_0\cdot 1+\epsilon_1\cdot 2^1+\epsilon_2\cdot 2^2+\dots}},$$

where each $\epsilon_i \in \{0,1\}$ and all but finitely many of the ϵ_i are zero. Now *every* nonnegative integer n has a unique expansion in the form

$$\epsilon_0\cdot 1+\epsilon_1\cdot 2^1+\epsilon_2\cdot 2^2+\dots;$$

this is just the binary expansion of n. Hence, the value of the infinite product is precisely $\sum_{n=0}^{\infty}\frac{1}{2^n}=2$.

Problem 4. In "shift geometry", a line shifts vertically 2 units as it crosses the y-axis, then continues with the same slope. For example, the "line" from $(-1,-2)$ to $(1,2)$ is as shown.



Where does the "shifted" line from $(-1,1)$ to $(2,-1)$ intersect the line shown?

(A) they do not intersect (B) $(\frac{4}{5},-\frac{1}{5})$
(C) $(-\frac{2}{5},\frac{3}{5})$ (D)$^\heartsuit$ $(\frac{2}{7},\frac{9}{7})$, (E) $(\frac{1}{8},\frac{1}{4})$

Solution. Lines in this geometry are of the form

$$y=\begin{cases} mx+b & \text{if } x\le 0,\\ mx+b+2 & \text{if } x>0.\end{cases}$$

Substituting $(-1,-2)$ and $(1,2)$ for (x,y) gives

$$\begin{aligned} -2 &= -m+b\\ 2 &= m+b+2.\end{aligned}$$

So for the given line $m=1$ and $b=-1$:

$$y=\begin{cases} x-1 & \text{if } x\le 0,\\ x+1 & \text{if } x>0.\end{cases}$$

Similarly the second line is described by

$$y=\begin{cases} -\frac{4}{3}x-\frac{1}{3} & \text{if } x\le 0,\\ -\frac{4}{3}x+\frac{5}{3} & \text{if } x>0.\end{cases}$$

These clearly do not intersect when $x \leq 0$, and when $x > 0$ they intersect when $-\frac{4}{3}x + \frac{5}{3} = x + 1$. So $x = \frac{2}{7}$ and $y = \frac{9}{7}$.

Problem 5. If a_n is defined recursively by $a_1 = 1$ and $a_{n+1} = \frac{1}{3}a_n$ for $n \geq 1$, then $a_n \to 0$ as $n \to \infty$. Find a value of c so that, if b_n is defined recursively by

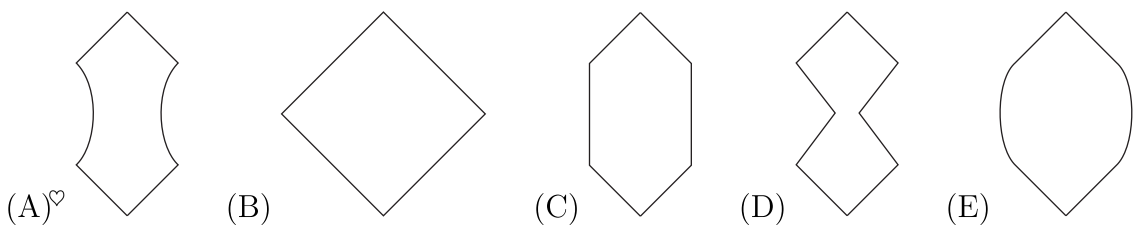
$$b_n = 1, \quad b_{n+1} = \frac{1}{3}b_n + c \quad \text{for } n \geq 1,$$

then $b_n \to 2014$ as $n \to \infty$.

(A)$^\heartsuit$ $\frac{4028}{3}$ (B) $\frac{2014}{3}$ (C) $\frac{1007}{3}$ (D) 2014 (E) there is no such c

Solution. Rewrite the recursive definition in the form $b_{n+1} - k = \frac{1}{3}(b_n - k)$. This is the same as $b_{n+1} = \frac{1}{3}b_n + c$ if and only if $k = \frac{3}{2}c$. Then $b_n - k \to 0$ as $n \to \infty$; i.e., $b_n \to k$. For $k = 2014$, we need $c = \frac{2}{3} \cdot 2014 = \frac{4028}{3}$.

Problem 6. Suppose you rotate a cube rapidly around one of its diagonals. Which of the following most closely resembles the silhouette of the resulting solid of revolution?



Solution. Try it! In fact, we did at the awards ceremony, using a cube that was color coded to make the middle section clear, and which doubled as the trophy for the top performers in this year's tournament. For pictures, or to 3-D print your own cube, go to

`http://www.thingiverse.com/thing:534801`

Alternatively, suppose you rotate around the diagonal AB. The top and bottom of the silhouette are determined by the edges emanating from the vertices A and B. These form cones. The edges not adjacent to A or B are skew to the diagonal, so their rotation determines a hyperboloid of one sheet. In particular this shows that the hyperboloid of one sheet is a ruled surface. In fact, the hyperboloid is doubly ruled, since there are two non-parallel edges not adjacent to A or B.

Problem 7. Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial of degree n. We call f **eponymous** if $f(0) = a_0, f(1) = a_1, \ldots, f(n) = a_n$. (Of course, $f(0) = a_0$ is true for all polynomials; the other conditions are typically not true.) Now suppose $f(x)$ is an eponymous polynomial of degree 2 and $f(0) = 1$. Find $f(3)$.

(A)$^\heartsuit$ -5 (B) -3 (C) -1 (D) 1 (E) 3

Solution. Since $f(0) = 1$, we know $f(x) = a_2x^2 + a_1x + 1$. The condition $f(k) = a_k$ implies

$$\begin{aligned} 4a_2 + 2a_1 + 1 &= a_2, \\ a_2 + a_1 + 1 &= a_1, \end{aligned}$$

from which we see $a_2 = -1$ and $a_1 = 1$. So $f(x) = -x^2 + x + 1$ and so $f(3) = -5$.

In general, if $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is eponymous of degree n, then it must satisfy $k^na_n + k^{n-1}a_{n-1} + \cdots + ka_1 + a_0 = a_k$ for all $0 \le k \le n$. With $\mathbf{v} = [a_0, \ldots, a_n]^T$, these conditions amount to the matrix equation

$$\begin{pmatrix} n^n & \ldots & n & 1 \\ \vdots & \ddots & \vdots & \vdots \\ k^n & \ldots & k & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \ldots & 1 & 1 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \mathbf{v} = \mathbf{v},$$

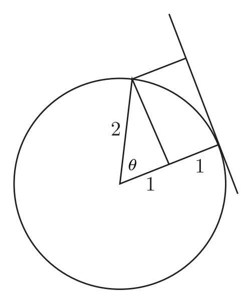
i.e., $\mathbf{v}$ is an eigenvector of the matrix on the left-hand side, with eigenvalue 1.

Definition of EPONYMOUS: `of, relating to, or being the person or thing for whom or which something is named`
(from `http://www.merriam-webster.com/dictionary/eponymous`)
Also the title of the album released in 1988 by famous Athens alternative rock band R.E.M.

Problem 8. Erect a pole of length 1 on a sphere of radius 2 in any direction you like, not necessarily perpendicular to the surface. Now shine a light so that the shadow of the pole on the sphere is as long as possible. What is the maximum possible length of the shadow?

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D)$^\heartsuit$ $\frac{2\pi}{3}$ (E) π

Solution. The longest shadow occurs when the light is perpendicular to the top of the pole and tangent to the sphere. The diagram shows then that $\theta = \arccos(\frac{1}{2}) = \frac{\pi}{3}$, so $s = r\theta = \frac{2\pi}{3}$.



Problem 9. Tetra and Yuri both want to approximate $\sum_{n=1}^{2014} \frac{1}{n}$. Tetra approximates each $\frac{1}{n}$ by rounding it up to the nearest $\frac{1}{10}$; for example, she approximates $\frac{1}{3}$ as 0.4.

Yuri approximates each $\frac{1}{n}$ by rounding down to the nearest $\frac{1}{10}$. What is the difference of their sums?

(A)$^\heartsuit$ 201 (B) 201.4 (C) 2010 (D) 2014 (E) ln(2014)

Solution. Notice that neither of them rounds $1, \frac{1}{2}, \frac{1}{5}$, or $\frac{1}{10}$. For the remaining 2010 terms their approximations differ by exactly 0.1, so the difference of their sums is $2010 \cdot 0.1 = 201$.

Problem 10. Suppose that x is chosen uniformly at random from the interval $(0, 1)$. What is the probability that the leftmost decimal digit of $\frac{1}{x}$ is 1?

(A) $\frac{2}{3}$ (B)$^\heartsuit$ $\frac{5}{9}$ (C) $\frac{1}{2}$ (D) $\frac{1}{9}$ (E) $\frac{1}{10}$

Solution. The leftmost digit of $\frac{1}{x}$ is $1 \iff \frac{1}{x} \in [1, 2) \cup [10, 20) \cup [100, 200) \cup \ldots \iff x \in (\frac{1}{2}, 1) \cup (\frac{1}{20}, \frac{1}{10}) \cup (\frac{1}{200}, \frac{1}{100}) \cup \ldots$. The sum of the lengths of these final intervals is $\frac{1}{2} + \frac{1}{20} + \frac{1}{200} + \cdots = \frac{1}{2}\sum_{k=0}^{\infty}(\frac{1}{10})^k = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{10}} = \frac{5}{9}$.

2 Medium Problems

Problem 11. What is the largest integer n for which $\dfrac{20!}{1!2!3!4!n!}$ is an integer?

(A) 5 (B) 10 (C) 11 (D) 13 (E)$^\heartsuit$ 15

Solution. Since $1+2+3+4+n = 20$ when $n = 10$, the expression is an integer for $n \le 10$. In fact, when $n = 10$ we get the multinomial coefficient $\binom{20}{1,2,3,4,10}$, which is the coefficient of $vw^2x^3y^4z^{10}$ in $(v+w+x+y+z)^{20}$.

Clearly 11 divides $\frac{20!}{1!\cdot 2!\cdot 3!\cdot 4!}$, since the numerator contains a factor of 11 but the denominator does not, and similarly for $13, 17, 19$. In fact,

$$20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19,$$

and so

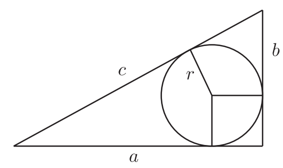
$$\frac{20!}{1!2!3!4!} = 2^{13} \cdot 3^6 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19.$$

Since $15! = 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ and $16! = 2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^3 \cdot 11 \cdot 13$, we have $n = 15$.

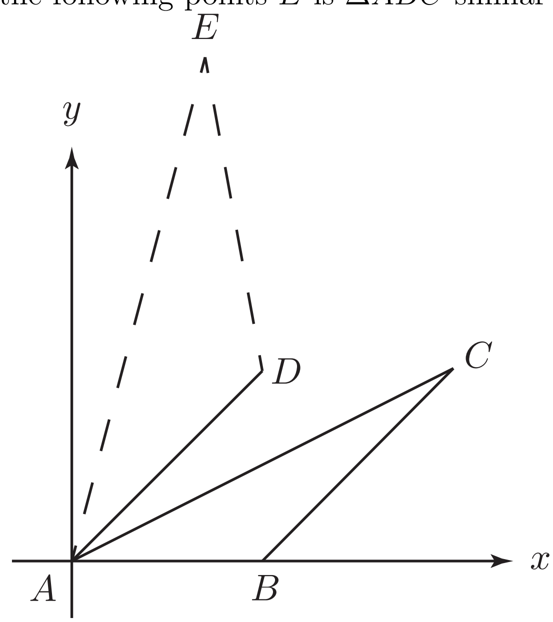
Problem 12. A **Pythagorean triangle** is one with integer side lengths a, b, c satisfying $a^2 + b^2 = c^2$. What is the smallest positive integer that does *not* occur as the radius of a circle inscribed in a Pythagorean triangle?

(A) 2 (B) 3 (C) 4 (D) 5 (E)$^\heartsuit$ every positive integer occurs

Solution. We need the following fact: If a circle of radius r is inscribed in an (a, b, c) Pythagorean triangle, then $r = \frac{a+b-c}{2}$. Taking $(a, b, c) = (3, 4, 5)$ shows that the radius 1 occurs. Since we can scale the entire figure by any integer factor, every positive integer occurs as a value of r.



Problem 13. Suppose $A = (0, 0)$, $B = (2, 0)$, $C = (4, 2)$, and $D = (2, 2)$. For which of the following points E is ΔABC similar to ΔADE?



(A) $(2, 4)$ (B) $(2, 5)$ (C)$^\heartsuit$ $(2, 6)$ (D) $(1, 4)$ (E) $(3, 5)$

Solution. Represent B, C, and D as complex numbers: $B = 2$, $C = 4+2i$, $D = 2+2i$. Then recall that multiplication by a complex number may rotate and change length, but does not change angles; i.e., it sends triangles to similar (though not necessarily congruent) triangles. In particular, multiplying by $1 + i$ carries A to A, B to D and C to $(4 + 2i)(1 + i) = 4 + 6i - 2 = 2 + 6i$. Thus, $E = (2, 6)$.

Problem 14. A square formation of Army cadets, 50 feet on the side, is marching forward at a constant pace. The company mascot, a bulldog, starts at the center of the rear rank, trots forward in a straight line to the center of the front rank, then trots back again in a straight line to the center of the rear. At the instant he returns to his original position, the cadets have advanced exactly 50 feet. Assuming the dog trots at a constant speed and loses no time in turning, how many feet does he travel?

(A) 50 ft (B)$^\heartsuit$ $50+50\sqrt{2}$ ft (C) 100 ft (D) $100+50\sqrt{2}$ ft (E) 150 ft

Solution. Without loss of generality, assume the troops are moving forward at a constant rate of 50 ft/min. Let v be the dog's constant speed (in ft/min) and t the time (in min) it takes him to reach the front. Then

$$vt = 50t + 50 \quad \text{or} \quad v = 50\frac{t+1}{t}$$

and

$$vt - v(1-t) = 50 \quad \text{or} \quad v(2t-1) = 50.$$

The first equation says that the distance the dog moves in t minutes (namely, vt) is equal to the distance the troops move ($50t$ ft) plus 50 ft (to move the dog from the back to the front). The second equation says, from the position reached at time t (namely, vt), the dog moves *backwards* for the remaining $1-t$ minutes, and ends up 50 feet ahead of his starting point.

Combining the two equations gives

$$50\left(\frac{t+1}{t}\right) \cdot (2t-1) = 50 \Rightarrow (t+1)(2t-1) = t$$
$$\Rightarrow (2t^2-1) = 0 \Rightarrow t = \frac{1}{\sqrt{2}};$$

hence,

$$v = 50\frac{\frac{1}{\sqrt{2}}+1}{\frac{1}{\sqrt{2}}} = 50(1+\sqrt{2}).$$

Problem 15. Fran the Frog is resting at Lilypad #0 in the middle of an infinite, bidirectional sequence of Lilypads numbered with the integers. Fran has the ability to jump forwards or backwards, but can only move by a square number of steps at each jump. For example, in 3 moves, Fran could jump 100 lilypads forward to 100, then 144 lilypads backward to -44, then 64 lilypads forward to 20. But this is not the shortest way of reaching 20, since Fran could have jumped forward 16 lilypads and then another 4 lilypads to get to 20 in just 2 moves.

If Fran wants to reach 2014 instead of 20, what is the smallest number of moves Fran can make?

(A) 2 (B)$^\heartsuit$ 3 (C) 4 (D) 5 (E) more than 5

Solution. The answer is 3. We first show that we can reach *any* Lilypad in three moves and then that 2014 actually requires three.

Jumping $+k^2$ steps and then $-(k-1)^2$ steps, we reach $k^2-(k-1)^2 = 2k-1$. So any odd numbered Lilypad can be reached in two steps. Any even numbered Lilypad can then be reached by an extra step of length 1.

We now show that 2014 needs 3 steps. Since 2014 is not a square, we need more than one step. To rule out two steps, we must demonstrate that 2014 cannot be written in the form x^2-y^2 or x^2+y^2, for any integers x and y. If $2014 = x^2-y^2$, then $2014 = (x-y)(x+y)$, and the two right-hand factors have the same parity.

Clearly, at least one must be even, since the product 2014 is even. But then both are even, forcing the product to be a multiple of 4, which 2014 is not. If $2014 = x^2 + y^2$, either x and y are both even or both odd. Hence, $\frac{x-y}{2}$ and $\frac{x+y}{2}$ are both integers, and

$$\left(\frac{x-y}{2}\right)^2 + \left(\frac{x+y}{2}\right)^2 = \frac{x^2+y^2}{2} = 1007,$$

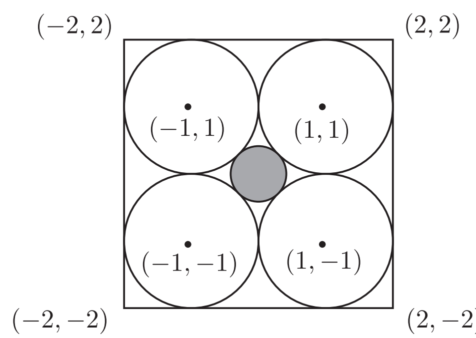
so that 1007 is the sum of two squares of integers. But this is impossible, since $1007 \equiv 3 \pmod 4$, while the square of an integer is 0 or 1 modulo 4.

Remark: If we replace "square number of steps" with "cube number of steps", one can show by fairly simple arguments that some Lilypads require at least 4 steps to reach and that no Lilypad requires more than 5. But it is an open problem to decide whether any Lilypad actually requires 5. For more on this theme, look up the **easier Waring problem**.

Problem 16. In $\mathbb{R}^n$, draw spheres of radius 1 at each of the points $(\pm 1, \pm 1, \ldots, \pm 1)$. Notice that each of these 2^n spheres is tangent to the adjacent spheres, and also tangent to (but contained in) the cube with vertices $(\pm 2, \pm 2, \ldots, \pm 2)$. Now draw one more sphere, centered at the origin, with the largest radius subject to the condition that no point of the other 2^n spheres is inside the central sphere. What is the smallest value of n for which the central sphere extends outside the cube?

(A) 5 (B) 8 (C) 9 (D)$^\heartsuit$ 10 (E) the central sphere never extends outside the cube

Solution. The center of any (non-central) sphere is $\sqrt{n}$ units from the origin, so the central sphere will meet it tangentially at a point $\sqrt{n} - 1$ units from the origin. So the central sphere has radius $\sqrt{n} - 1$. When $n = 9$, this radius is 2, so the central sphere is tangent to the cube. For $n \geq 10$, it extends beyond the cube.



Problem 17. What is the smallest value of n so that $\sum_{k=1}^{n} \arctan(k) \geq 2\pi$.

(A) 4 (B) 5 (C)$^\heartsuit$ 6 (D) 7 (E) 8

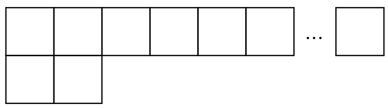
Solution. Note that $\arctan(1) = \frac{\pi}{4}$ and that $\frac{\pi}{4} < \arctan(k) < \frac{\pi}{2}$ whenever $k > 1$. Thus, n must be greater than 4 and no more than 8. The angle $\arctan(k)$ is the angle between the x-axis and the vector $(1, k)$; equivalently, it is the argument of the complex number $1 + ik$. Since the argument of the product of complex numbers is the sum of their arguments, one can think of $\sum_{k=1}^{n} \arctan(k)$ as the argument of

$$(1+i)(1+2i)(1+3i)\cdots(1+ni).$$

Now start multiplying:

$$(1+i)(1+2i) = -1+3i,$$
$$(-1+3i)(1+3i) = -10.$$

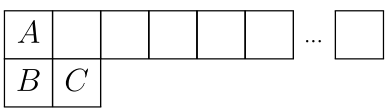
This shows that $\sum_{k=1}^{3} \arctan(k) = \pi$. Therefore, $\sum_{k=1}^{5} \arctan(k) < 2\pi$ (since each term in the sum is less than $\frac{\pi}{2}$) and $\sum_{k=1}^{6} \arctan(k) > 2\pi$ (since each of the first three angles is less than each of the last three). So $n = 6$.

Problem 18. 

Suppose a total of n squares are arranged as shown, $n \geq 4$. In how many ways can the numbers $1, 2, \ldots, n$ can be placed in the box so that both rows are increasing left to right, and all columns are increasing top to bottom?

(A)$^{\heartsuit}$ $\frac{n(n-3)}{2}$ (B) $\frac{n^2-5n+8}{2}$ (C) $\frac{n(n-1)}{2}$ (D) $\frac{n^2-3n+2}{2}$ (E) none of the above

Solution.



A must be 1. C must be at least 4, since there have to be numbers less than C to its left and above. For any choice of C, we must have $1 < B < C$, so there are $C-2$ choices for B. Once A, B, and C are in place, everything else is determined. So the total number of arrangements is

$$\begin{aligned}\sum_{C=4}^{n}(C-2) &= \sum_{C=4}^{n} C - 2(n-3)\\ &= \frac{n(n+1)}{2} - (1+2+3) - 2(n-3)\\ &= \frac{n^2-3n}{2}.\end{aligned}$$

Remark: Recall from the solution to Ciphering Problem #7 that the collection of symmetries of an object forms an algebraic structure called a **group**. Mathematicians also study the ways groups act on algebraic structures called **vector spaces**. This is the foundation of **Representation Theory**, a topic that spans many areas of mathematics and which has applications to modern chemistry and physics. It is a known fact that the representations of the symmetric group S_n are parametrized by the partitions of n, and they can be studied using **Young Diagrams**. These are the diagrams you saw above and in Ciphering Problem #2. The instructions you were given about labeling these diagrams correspond to a **standard labeling** for a Young

Diagram. Counting the number of standard labelings gives information about the dimension of the representation that corresponds to that diagram.

Problem 19. Evaluate the product $\prod_{k=1}^{45}(1+\tan(k^\circ))$.

(A) $(\frac{1+\sqrt{5}}{2})^{23}$ (B) 3^{15} (C) $2^{45/2}$ (D) π^{14} (E)$^\heartsuit$ none of the above

Solution. Observe that for each $\theta = 0^\circ, 1^\circ, 2^\circ, \ldots, 45^\circ$,

$$\begin{aligned}
&(1+\tan(\theta))(1+\tan(45^\circ-\theta)) = 1+\tan(\theta)+\tan(45^\circ-\theta)+\tan(\theta)\tan(45^\circ-\theta) \\
&\quad = 1 + \frac{\tan(\theta)+\tan(45^\circ-\theta)}{1-\tan(\theta)\tan(45^\circ-\theta)}(1-\tan(\theta)\tan(45^\circ-\theta)) + \tan(\theta)\tan(45^\circ-\theta) \\
&\quad = 1+\tan(45^\circ)(1-\tan(\theta)\tan(45^\circ-\theta))+\tan(\theta)\tan(45^\circ-\theta) \\
&\quad = 1+(1-\tan(\theta)\tan(45^\circ-\theta))+\tan(\theta)\tan(45^\circ-\theta) \\
&\quad = 2.
\end{aligned}$$

After inserting a harmless extra factor of $1+\tan(0)=1$, we can write

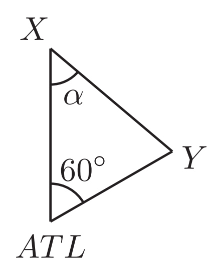
$$(1+\tan(1^\circ))\cdots(1+\tan(45^\circ)) = \prod_{\theta=0}^{22}\big((1+\tan(\theta))(1+\tan(45^\circ-\theta))\big) = 2^{23}.$$

Problem 20. Consider a collection of airports at distinct distances from each other. A plane leaves each airport and flies to the nearest other airport. What is the most planes that could land at the same airport?

(A) 4 (B)$^\heartsuit$ 5 (C) 6 (D) 7 (E) 8

Solution. Notice that 5 airports, equally spaced on a circle centered at a sixth airport, say ATL, *almost* gives an example, if there are no other airports nearby. Now move those 5 airports radially towards ATL by small but differing amounts.

If we tried to arrange 6 airports around ATL, any two "adjacent" airports, say X and Y, and ATL, would form a triangle with a 60° angle at ATL. The other angles can't be 60°, since all distances are distinct, so one angle α must be greater than 60°. Then the side opposite α must be longer than the side opposite ATL, i.e., Y is closer to X than to ATL.



3 Hard problems

Problem 21. For how many nonnegative integers $m < 2014$ is the polynomial

$$1 + x^{2014} + x^{2\cdot 2014} + \cdots + x^{2014\cdot m}$$

evenly divisible by the polynomial

$$1 + x + \cdots + x^m?$$

(A) 1 (B)$^{\heartsuit}$ 936 (C) 1007 (D) 2013 (E) 2014

Solution. We claim that this divisibility holds for a given m precisely when $\gcd(m+1, 2014) = 1$. Suppose for now that the claim is proved. Writing $n = m + 1$, we are asked to count the number of positive integers $n \in [1, 2014]$ coprime to 2014. Now $2014 = 2\cdot 19\cdot 53$. An integer in $[1, 2014]$ is divisible by 2 with probability exactly $\frac{1}{2}$, and similarly for 19 and 53; moreover, these events are independent. Consequently, the probability an $n \in [1, 2014]$ is coprime to 2014 is precisely $(1-\frac{1}{2})(1-\frac{1}{19})(1-\frac{1}{53}) = \frac{936}{2014}$. Thus, the total number of these n is 936. (Fans of number theory will recognize that we have just computed $\phi(936)$, where ϕ is the Euler phi-function.)

Now we turn to the proof of the claim. Simplifying the geometric series, we have to determine when

$$\frac{1 - x^{m+1}}{1 - x} \quad \text{divides} \quad \frac{1 - x^{2014(m+1)}}{1 - x^{2014}},$$

or equivalently, when

$$\frac{1 - x^{2014(m+1)}}{1 - x^{2014}} \frac{1 - x}{1 - x^{m+1}}$$

simplifies to a polynomial. This occurs precisely when every complex root of the denominator $(1 - x^{2014})(1 - x^{m+1})$ appears to at least the same multiplicity in the numerator $(1 - x^{2014(m+1)})(1 - x)$.

Since $1 - x^{2014}$ and $1 - x^{m+1}$ both divide $1 - x^{2014(m+1)}$, every root of the denominator is a root of $1 - x^{2014(m+1)}$, and so also a root of the numerator. The root $z = 1$ always occurs with multiplicity 2 in both the numerator and denominator. Moreover, if $z \neq 1$ is a root of the numerator, then z is a root of $1-x^{2014(m+1)}$ and not $1-x$; since $1 - x^{2014(m+1)}$ has only simple roots, z appears with multiplicity 1 in the numerator. So in order for the denominator to divide the numerator, it is necessary and sufficient that every root $z \neq 1$ of $(1 - x^{2014})(1 - x^{m+1})$ appears with multiplicity 1.

Claim: This holds exactly when $m + 1$ and 2014 are relatively prime.

If $m+1$ and 2014 share a common factor $d > 1$, then $(1-x^d)^2 \mid (1-x^{2014})(1-x^{m+1})$, and so each dth root of unity $z \neq 1$ occurs with multiplicity larger than 1.

Now suppose $\gcd(m + 1, 2014) = 1$. Since $1 - x^{m+1}$ and $1 - x^{2014}$ have only simple roots, it suffices to show that their only common root is $z = 1$. The roots of $1 - x^{2014}$ are the numbers $e^{2\pi ik/2014}$, where $0 \le k < 2014$, and the roots of $1 - x^{m+1}$ are the numbers $e^{2\pi i\ell/(m+1)}$, where $0 \le \ell < m + 1$. Any common root z corresponds to a rational number that is both of the form $\frac{k}{2014}$ and $\frac{\ell}{m+1}$. When put in lowest terms, this rational number has denominator dividing both 2014 and $m + 1$. Since $\gcd(m + 1, 2014) = 1$, this denominator is 1, so our rational number is an integer. Since $0 \le k < 2014$, our rational number is 0, forcing $z = e^{2\pi i\cdot 0/2014} = 1$.

Problem 22. Evaluate

$$\sum_{m=1}^{100} \lfloor \cos^2\left(\pi \cdot \frac{(m-1)!+1}{m}\right) \rfloor.$$

Here $\lfloor x \rfloor$ is the *floor function* of x, i.e., the largest integer less than or equal to x.

(A) 1 (B) 12 (C) 25 (D)$^\heartsuit$ 26 (E) 100

Solution. Since $0 \le \cos^2(\theta) \le 1$, we see that $\lfloor \cos^2(\theta) \rfloor = 0$ except when $\cos(\theta) = \pm 1$, i.e., when $\theta = k\pi$ for some $k \in \mathbb{Z}$. Hence, the sum is counting those values of $m \le 100$ for which m divides $(m-1)!+1$. This obviously includes $m = 1$. A theorem of Wilson asserts that this divisibility holds for an integer $m > 1$ if and only if m is prime. Since there are 25 primes up to 100, the final answer is $1 + 25 = 26$.

Problem 23. Let $f(x) = 2x^2 - 1$, and let $f^{(k)}(x)$ denote the kth iterate of $f(x)$. That is, $f^{(0)}(x) = f(x)$ and $f^{(k+1)}(x) = f(f^{(k)}(x))$ for each nonnegative integer k. For how many distinct real values of t is $f^{(2014)}(t) = 1$?

(A) 1 (B) 3 (C)$^\heartsuit$ $2^{2013}+1$ (D) $2^{2014}-1$ (E) 2^{2014}

Solution. Observe that if $f(x) \in [-1, 1]$, then $2x^2 \in [0, 2]$, so that x itself lies in $[-1, 1]$. Iterating this observation, we see that all possible values of t with $f^{(2014)}(t) = 1$ satisfy $-1 \le t \le 1$. Write $t = \cos(\theta)$, where $0 \le t \le \pi$ is uniquely determined. Then $f(t) = 2\cos^2(\theta) - 1 = \cos(2\theta)$. Similarly, $f^{(2)}(t) = \cos(4\theta)$, $f^{(3)}(t) = \cos(8\theta)$ and in general $f^{(n)}(t) = \cos(2^n\theta)$. Consequently, $f^{(2014)}(t) = 1$ precisely when $2^{2014}\theta$ is an integer multiple of 2π. Since $\theta \in [0, \pi]$, we see $0 \le 2^{2014}\theta \le 2^{2014}\pi$. The number of multiples of 2π in $[0, 2^{2014}\pi]$ is precisely $2^{2013} + 1$.

Problem 24. Consider a triangle in the plane whose vertices have integer coordinates. Recall that Pick's Theorem says that the area of this triangle is

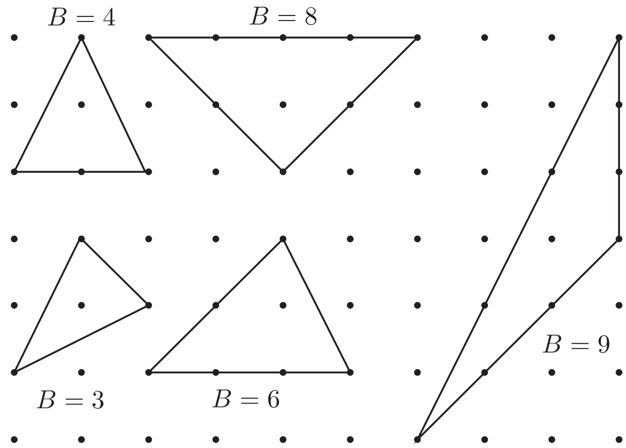
$$A = I + \frac{B}{2} - 1$$

where I is the number of integer points in the interior of the triangle and B is the number of integer points on the boundary. Notice that $B \ge 3$ always, since the 3 vertices of the triangle are integer points.

If $I = 1$, what is the largest B can be?

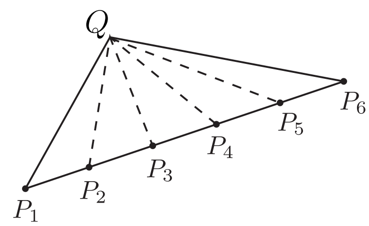
(A) 8 (B)$^\heartsuit$ 9 (C) 10 (D) 12 (E) more than 12

Solution. First some examples to show that B can be any of $3, 4, 6, 8$, or 9.



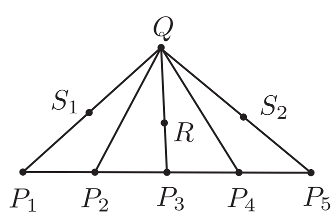
Claim: $B \leq 9$.

Proof. We first determine the maximum number of lattice points that can lie on an edge of such a triangle. Suppose 6 integer points $P_1, \ldots, P_6$ lie on the edge opposite vertex Q as shown:



The six points will be equally spaced, so that the 5 triangles ΔQP_iP_{i+1}, $i = 1, 2, \ldots, 5$, will all have the same area. Since there is exactly one interior point R in ΔQP_1P_6, at least one of the triangles ΔQP_2P_3, ΔQP_3P_4, ΔQP_4P_5, ΔQP_5P_6 has $B = 3$ and $I = 0$, so $A = \frac{1}{2}$. Moreover, at least one of them has an additional interior or boundary point, so that $A > \frac{1}{2}$. This contradiction shows that there cannot be 6 integer points on an edge.

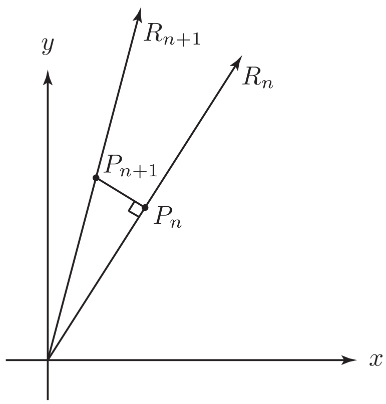
Suppose next that there are 5 integer points $P_1, \ldots, P_5$ on the edge opposite Q.



Divide into equal area triangles as above; this is only possible if R lies on edge QP_3, and thus edges QP_1 and QP_5 must each contain one lattice point. So if there are 5 lattice points on one side, the triangle must have exactly 8 lattice points on its boundary.

Any other Pick's triangle with one interior point must have ≤ 4 integer points per side – these are the examples drawn above, so $B \leq 9$. $\square$

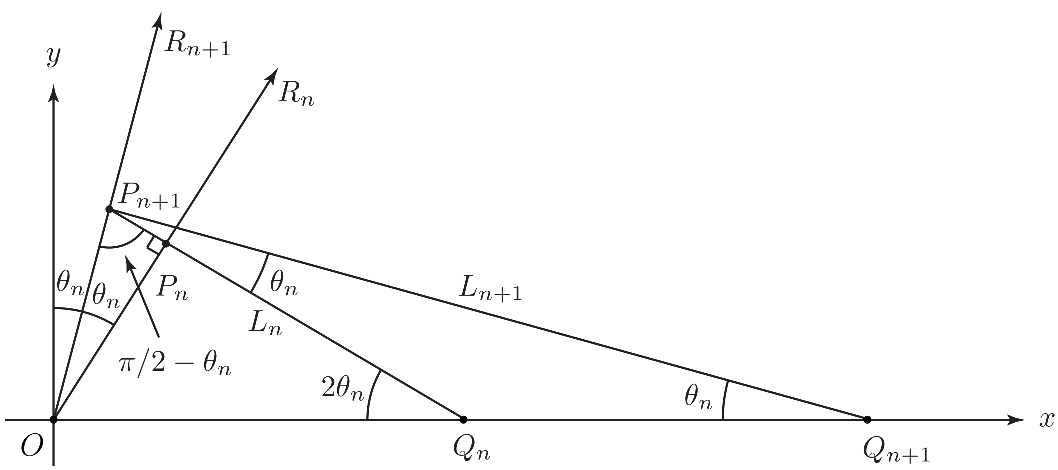
Problem 25. Let R_0 be the positive x-axis and $P_0 = (1, 0)$. Given R_n and P_n, let R_{n+1} be the ray in the first quadrant which bisects the angle between R_n and the positive y-axis, and let P_{n+1} be the intersection of R_{n+1} with the line through P_n perpendicular to R_n. The sequence of points $P_0, P_1, \ldots$ approaches the y-axis. What is the y-coordinate of the limit of that sequence?



(A) $\frac{1+\sqrt{5}}{2}$ (B) 2 (C) e (D) π (E)$^{\heartsuit}$ $\frac{\pi}{2}$

Solution. First let's name some things.

P_n and R_n are already defined. Let O be the origin. Let L_n be the line joining P_n and P_{n+1}, and let Q_n the point at which L_n intersects the x-axis. Let θ_n be the angle between R_n and R_{n+1}. θ_n shows up in many places; in particular, θ_n is also the angle between R_{n+1} and the positive y-axis.



For example, $P_1 = (1, 1)$, L_1 is the line $y = -x + 2$, $Q_1 = (2, 0)$ and $\theta_1 = \pi/8$.

Since $\theta_1 = \pi/8$, we have that $\theta_n = \pi/2^{n+2}$.

Now notice that $\Delta OP_{n+1}Q_n$ is isosceles, since both $\angle OP_{n+1}Q_n$ and $\angle P_{n+1}OQ_n$ measure $\pi/2 - \theta_n$. This implies that $\angle OQ_nP_{n+1}$ equals $2\theta_n$.

Next notice that $\Delta P_{n+1}Q_nQ_{n+1}$ is also isosceles, since both $\angle P_{n+1}Q_{n+1}Q_n$ and $\angle Q_nP_{n+1}Q_{n+1}$ equal θ_n. This implies that $|Q_nQ_{n+1}| = |Q_nP_{n+1}|$, which we know from the last paragraph to also equal $|OQ_n|$. Consequently, $|OQ_{n+1}| = 2 \cdot |OQ_n|$.

Since

$$|OQ_1| = 2, \quad \text{we obtain inductively that} \quad |OQ_n| = 2^n.$$

Finally we want to find $|OP_{n+1}|$. This is approximately the length of the circular arc subtended by O and P_{n+1} on the circle centered at Q_n. That arc has length $|OQ_n| \cdot 2\theta_n = 2^n \cdot \frac{\pi}{2^{n+1}} = \pi/2$. Since this is independent of n, it follows that

$$\lim_{n\to\infty} |OP_{n+1}| = \lim \frac{\pi}{2} = \frac{\pi}{2}.$$

Authors. Written by Mo Hendon, Paul Pollack, and Amber Russell. We thank Will Kazez for his help in drawing the pictures.

Sources. Problem #14 is taken from Martin Gardner's *My Best Mathematical and Logic Puzzles*, problem 35, pp. 18–19. Martin Gardner (1914–2010) was a popular science writer who (anong many other contributions) wrote the monthly "Mathematical Games" column for *Scientific American* from 1956 to 1981. While having no formal training in mathematics, his writings served to encourage an entire generation of budding young scientists. For more information on Gardner, see

`http://en.wikipedia.org/wiki/Martin_Gardner`

Problem #20 was posted by Tatiana Shubin on `http://www.mathteacherscircle.org`. Problem #21 is adapted from Problem #1 on the 1977 USAMO. Problem #23 is adapted from a 1998 competition in Turkey. Problem #25 was inspired by retired UGA Math Professor Roy Smith, who offered the given solution as a more geometric approach to Problem #25 on the 2013 UGA HSMT written test.