Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes
November 16, 2013

WITH SOLUTIONS

1 Easy Problems

Problem 1. Find $\prod_{n=2}^{2013}\left(1-\frac{1}{n}\right)$. The notation "$\prod_{n=1}^{k} a_n$" means $a_1 \cdot a_2 \cdot a_3 \cdots a_k$.

(A) $1-\frac{1}{2014}$ (B)$^\heartsuit$ $\frac{1}{2013}$ (C) $1+\frac{1}{2013}$ (D) $\frac{1}{2013!}$ (E) $1-\frac{1}{2013!}$

Solution. $\prod_{n=2}^{2013}\left(1-\frac{1}{n}\right) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{2013}\right) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{2012}{2013} = \frac{1}{2013}$.

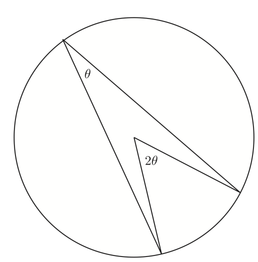
Problem 2. If a triangle with a $30°$ angle is inscribed in a circle of radius 1, how long is the side opposite the $30°$ angle?

(A)$^\heartsuit$ 1 (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\sqrt{3}$ (E) it depends on the other angles in the triangle

Solution. The interesting thing here is that it doesn't matter what the other angles of the triangle are, so we may as well assume that it's a $30° - 60° - 90°$ right triangle. Then the hypotenuse is a diameter, and thus length 2, and so opposite side length $= 2\sin(30°) = 1$.

Why does it not matter what the other angles are? This follows from the familiar fact that the angle subtended by an arc from a point on a circle is **half** the angle subtended by the same arc from the center.

Alternatively, recall the law of sines says



$$\frac{A_1}{\sin(\alpha_1)} = \frac{A_2}{\sin(\alpha_2)} = \frac{A_3}{\sin(\alpha_3)},$$

if A_i is the side opposite angle α_i, $i = 1, 2, 3$. This common ratio is the diameter of the circumscribed circle, so again

$$2 = \frac{A_1}{\sin(30°)} \Rightarrow A_1 = 1.$$

Problem 3. If you travel a certain distance at a rate of R_1 miles/hr, and the same distance again at a rate of R_2 miles/hr, what is your average rate in miles per hour for the whole trip?

(A) $\frac{R_1+R_2}{2}$ (B) $\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$ (C) $\frac{1}{2}\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$ (D)$^\heartsuit$ $\frac{2}{\frac{1}{R_1}+\frac{1}{R_2}}$ (E) None of the above

Solution. Your time traveling the initial distance d is $t_1 = \frac{d}{R_1}$, and the time for the second distance (also d) is $t_2 = \frac{d}{R_2}$. So the combined rate is $\frac{2d}{t_1+t_2} = \frac{2d}{\frac{d}{R_1}+\frac{d}{R_2}} = \frac{2}{\frac{1}{R_1}+\frac{1}{R_2}}$.

Problem 4. What is the smallest positive integer n so that the leftmost digit of $(11)^n$ is **not** 1?

(A) 7 (B)$^\heartsuit$ 8 (C) 9 (D) 10 (E) there is no such integer

Solution. If you just start computing powers of 11, you may notice Pascal's triangle arising: $11^2 = 121$, $11^3 = 1331$, $11^4 = 14641$. Of course this fails for 10^5, because we need to account for the carries:

$$\begin{aligned} 11^5 &= 1 \times 10^5 + 5 \cdot 10^4 + 10 \times 10^3 + 10 \times 10^2 + 5 \times 10^1 + 1 \\ &= 1 \times 10^5 + 6 \times 10^4 + 1 \times 10^3 + 0 \times 10^2 + 5 \times 10^1 + 1 = 161051. \end{aligned}$$

Compare with the 5th row of Pascal's triangle, and you'll see how the carries work:

$$1 \quad 5 \; 10 \; 10 \; 5 \; 1.$$

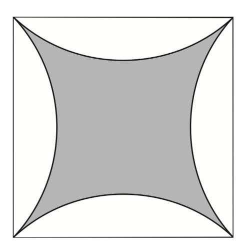
This makes it clear that 11^{10} does not begin with a 1, since the 10th row of Pascal's triangle begins 1 10 45 $\ldots$, and so 11^{10} will begin with a 2. What about smaller exponents? Look at the left part of Pascal's triangle:

$$\begin{array}{ccccccc} & & & 1 & 5 & 10 & \ldots \\ & & 1 & 6 & 15 & 20\ldots & \\ & 1 & 7 & 21 & 35 & \ldots & \\ 1 & 8 & 28 & 56 & \ldots & & \end{array}$$

So $11^7 = 194\ldots$ and $11^8 = 21\ldots$.

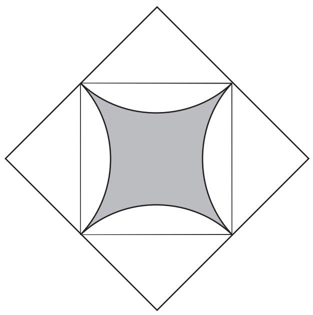
Why does 11^n show up in Pascal's triangle? Write 11^n as $(10+1)^n$ and apply the binomial theorem.

Problem 5. In the diagram shown at right, 4 circular arcs pass through the corners of a 1 by 1 square. Each of the arcs is tangent to the diagonals of the square. What is the area of the shaded region?



(A) $1-\frac{\pi}{4}$ (B) $1+\sqrt{3}-\frac{2\pi}{3}$ (C)$^{\heartsuit}$ $2-\frac{\pi}{2}$ (D) $4-\pi$ (E) $\frac{\pi}{2}-1$

Solution. See the diagram:



The larger square has edge length $\sqrt{2}$, so its area is 2. The unshaded part now consists of 4 quarter circles of radius $\frac{\sqrt{2}}{2}$, so its area is $\frac{\pi}{2}$. So the shaded area is $2-\frac{\pi}{2}$.

Problem 6. What is the smallest natural number r so that no number of the form $n!$ ends in exactly r zeroes?

(A) 4 (B)$^{\heartsuit}$ 5 (C) 6 (D) 10 (E) 25

Solution. The number of zeroes at the end of $n!$ equals the number of factors of 5. So 5! has 1 zero, 10! has 2 zeros, 15! has 3, 20! has 4 but 25! has 6. Therefore no $n!$ ends in 5 zeroes.

Problem 7. If $\tan(x)+\tan(y)=2013$ and $\tan(x+y)=2014$, what is $\cot(x)+\cot(y)$?

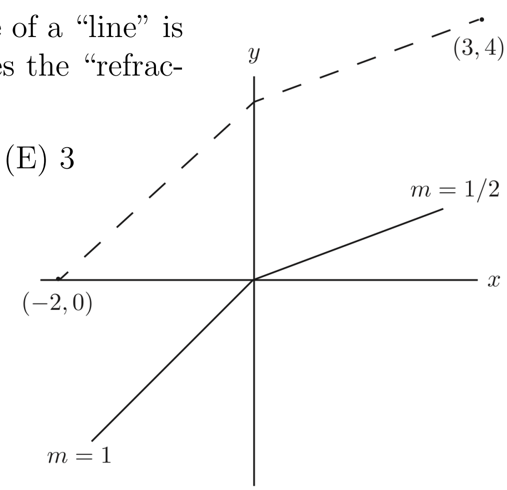
(A) $\frac{1}{2013}$ (B) $\frac{2013}{2014}$ (C)$^{\heartsuit}$ $2013\cdot 2014$ (D) 2013 (E) 2014

Solution. $\tan(x+y)=2014 \Rightarrow \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}=2014 \Rightarrow \frac{2013}{2014}=1-\tan(x)\tan(y) \Rightarrow \tan(x)\tan(y)=\frac{1}{2014}$. Thus,

$$\cot(x)+\cot(y)=\frac{1}{\tan(x)}+\frac{1}{\tan(y)}=\frac{\tan(x)+\tan(y)}{\tan(x)\tan(y)}=\frac{2013}{1/2014}=2013\cdot 2014.$$

Problem 8. In "refraction geometry", the slope of a "line" is halved as the line crosses the y-axis. Where does the "refraction line" from $(-2,0)$ to $(3,4)$ cross the y-axis?

(A) 1 (B) 8/5 (C) 2 (D)$^{\heartsuit}$ 16/7 (E) 3



Solution. Lines in this geometry are of the form

$$y = \begin{cases} mx + b & \text{if } x \le 0, \\ \frac{1}{2}mx + b & \text{if } x > 0 \end{cases}$$

Substituting $(-2, 0)$ and $(3, 4)$ for (x, y) gives

$$\begin{aligned} 0 &= -2m + b, \\ 4 &= \frac{3}{2}m + b. \end{aligned}$$

Solving, we get $m = \frac{8}{7}$ and $b = \frac{16}{7}$.

We call this "refraction geometry" because it mimics (not exactly – see Snell's law) how light bends when it passes from one medium to another.

Problem 9. Suppose that n is the largest integer for which 3^n divides the number

$$12345678910111121314\ldots2013$$

obtained by concatenating the decimal digits of the positive integers $1, 2, 3, \ldots, 2013$. Find n.

(A) 0 (B)$^\heartsuit$ 1 (C) 2 (D) 3 (E) None of the above

Solution. We recall that every positive integer is congruent to its sum of digits modulo 9. Applying this fact **twice**, we see that

$$\begin{aligned} 12345678910111121314\ldots2013 &\equiv 1 + 2 + 3 + \cdots + 2 + 0 + 1 + 3 \pmod 9 \\ &\equiv 1 + 2 + 3 + \cdots + 2013 \pmod 9). \end{aligned}$$

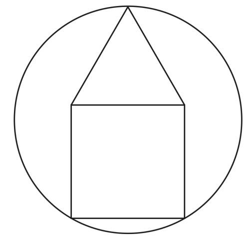
(The right-hand summands on the first line are the individual digits of the numbers $1, 2, \ldots, 2013$, and those on the second line are the numbers $1, 2, \ldots, 2013$ themselves.) Since

$$1 + 2 + 3 + \cdots + 2013 = \frac{2013 \cdot 2014}{2} = 2013 \cdot 1007 \equiv 6 \cdot 8 \equiv 3 \pmod 9,$$

we see that 3 divides our number but that 9 does not. So the answer is $n = 1$.

The real number $0.123456789101112\ldots$ is known as *Champernowne's constant*. That constant is known to be irrational (in fact transcendental) and is also **normal**: every finite string of decimal digits of a given length shows up in the decimal expansion with the same frequency as every other string of the same length. In fact, this constant was the first explicit real number to be proved normal. This was done by D. Champernowne, an English economist and mathematician, while he was still an undergraduate.

Problem 10. A square of side length 1 is topped with an equilateral triangle, then inscribed in a circle as shown. What is the radius of the circle?



(A)$^\heartsuit$ 1 (B) $\frac{1}{2}+\frac{\sqrt{3}}{4}$ (C) $\frac{3}{2}$ (D) $2+\sqrt{3}$ (E) 2

Solution. Slide the triangle to the bottom of the square. Notice that the top point of the triangle has moved down 1 unit, and is 1 unit from the bottom corners of the square.

2 Medium Problems

Problem 11. How many positive integers n are there for which $\lfloor \frac{n^2}{5} \rfloor$ is a prime number? Here the notation $\lfloor x \rfloor$ means the largest integer not exceeding x.

(A) 2 (B)$^\heartsuit$ 3 (C) 4 (D) 5 (E) infinitely many

Solution. We consider 5 cases:

- $n = 5k$. Then $\lfloor n^2/5 \rfloor = 5k^2$, which is prime only if $k = 1$.
- $n = 5k+1$. Then $\lfloor (5k+1)^2/5 \rfloor = 5k^2+2k = (5k+2)k$. This is prime only if $k = 1$.
- $n = 5k+2$. Then $\lfloor (5k+2)^2/5 \rfloor = 5k^2+4k = (5k+4)k$, which is never prime.
- $n = 5k+3$. Then $\lfloor (5k+3)^2/5 \rfloor = 5k^2+6k+1 = (5k+1)(k+1)$, which is never prime.
- $n = 5k+4$. Then $\lfloor (5k+4)^2/5 \rfloor = 5k^2+8k+3 = (5k+3)(k+1)$, which is prime only if $k = 0$.

So the only primes are when $n = 5, 6$, and 4.

Problem 12. Let $\theta_1, \theta_2, \theta_3, \theta_4$ be the four complex roots of the polynomial

$$P(x) = 20x^4 + 13x^3 + 11x + 16.$$

Find the numerical value of

$$(1+\theta_1)(1+\theta_2)(1+\theta_3)(1+\theta_4).$$

(A) 16 (B) -13 (C) 12 (D) $\frac{4}{5}$ (E)$^{\heartsuit}$ $\frac{3}{5}$

Solution. Notice that $1+\theta_1, \ldots, 1+\theta_4$ are the roots of the polynomial $Q(x) = P(x-1)$. $Q(x)$ also has leading coefficient 20, so the product of its roots is $1/20$ of the constant term of $Q(x)$, i.e., $Q(0)/20$. But $Q(0) = P(0-1) = 12$, so the product is $12/20 = 3/5$.

Problem 13. The function $B(n)$ satisfies

$$B(0) = 0, \quad B(2n) = B(n) \quad \text{and} \quad B(2n+1) = B(n) + 1$$

for every nonnegative integer n. Find $B(2013)$.

(A) 7 (B) 8 (C)$^{\heartsuit}$ 9 (D) 10 (E) 11

Solution. You can solve this with patience: $B(2013) = B(2\cdot 1006+1) = B(1006) + 1 = B(503) + 1 = \ldots$. Or you can notice that $B(n)$ counts the number of 1's in the binary expansion of n. This is true because if the binary expansion of n is $a_k \cdots a_0$, then the expansion of $2n$ is $a_k \ldots a_0 0$, and that of $2n+1$ is $a_k \cdots a_0 1$. So we need to find the binary expansion of 2013, which is 11111011101. So $B(2013) = 9$.

Problem 14. Consider a triangular array of numbers whose nth row, $1 \le n \le 100$, consists of the number n repeated n times. What is the average of all of the numbers in the array?

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 2 & & 2 & & & \\
 & & 3 & & 3 & & 3 & & \\
 & 4 & & 4 & & 4 & & 4 & \\
5 & & 5 & & 5 & & 5 & & 5 \\
\iddots & & \iddots & & & & \ddots & & \ddots \\
100 & & & & \cdots & & & & 100
\end{array}$$

(A) $\frac{101}{2}$ (B) $\frac{20301}{6}$ (C) 5050 (D) $\frac{2030100}{6}$ (E)$^{\heartsuit}$ $\frac{201}{3}$

Solution. First notice that the sum of the numbers in the nth row is n^2, so the sum of all numbers in the array is $\sum_{n=1}^{100} n^2 = \frac{100(101)(201)}{6}$. The number of numbers in the array is $\sum_{n=1}^{100} n = \frac{100(101)}{2}$. So the average is

$$\frac{100\cdot 101\cdot 201/6}{100\cdot 101/2} = \frac{201}{3}.$$

We have used two important formulas: $1+2+\cdots+N = \frac{N(N+1)}{2}$ and $1^2+2^2+\cdots+N^2 = \frac{N(N+1)(2N+1)}{6}$. What if you don't remember those formulas? You probably remember the proof of the first one:

$$\begin{array}{rrrrrrrrr} & 1 & + & 2 & + & \dots & + & N & = & x \\ + & N & + & (N-1) & + & \dots & + & 1 & = & x \\ \hline & N+1 & + & & & \dots & & N+1 & = & 2x. \end{array}$$

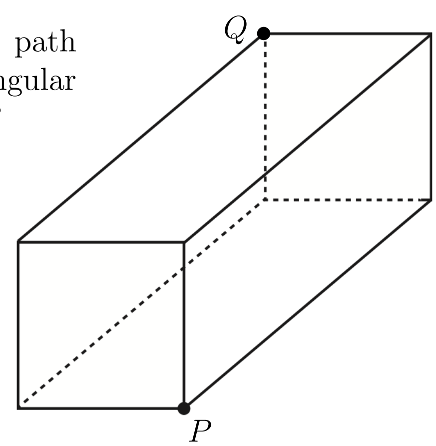
Here is a similar derivation for the second formula: Begin with the triangular array from this question, and take advantage of the symmetry; rotate the triangle (twice) by 120°:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & & 2 & & \\ & \iddots & & & & \ddots & \\ N & & N & & N & & N \end{array} \qquad \begin{array}{ccccccc} & & & N & & & \\ & & \iddots & & \ddots & & \\ & 2 & & & & \ddots & \\ 1 & & 2 & & \cdots & & N \end{array}$$

$$\begin{array}{ccccccc} & & & N & & & \\ & & \iddots & & \ddots & & \\ & \iddots & & & & 2 & \\ N & & \cdots & & 2 & & 1 \end{array}$$

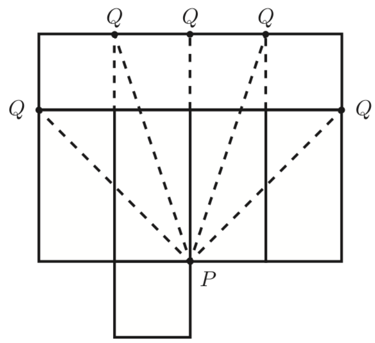
The sum of the corresponding entries in the three triangles is $2N+1$. So if x is the desired sum, $3x = (2N+1) \cdot \#\text{entries in a triangle} = (2N+1)\left(\frac{N(N+1)}{2}\right)$. So $x = \frac{N(N+1)(2N+1)}{6}$.

Problem 15. What is the length of the shortest path lying entirely on the surface of a 1 by 1 by 2 rectangular box, going from one corner to the opposite corner?



(A) $\sqrt{6}$ (B)$^{\heartsuit}$ $\sqrt{8}$ (C) $\sqrt{10}$ (D) $2+\sqrt{2}$ (E) 4

Solution. Look at a **net** — a way of cutting the box open and laying it flat, where we can use the usual tools of geometry. In fact, we'll look at several nets at once:



Notice that we've drawn different copies of the back square of the box — each with a point Q on it. This shows that the shortest path has length $\sqrt{8}$, and that there are two shortest paths.

Problem 16. Find the sum of the infinite series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \dots,$$

where the denominators are all of the positive integers which have no prime factors larger than 3.

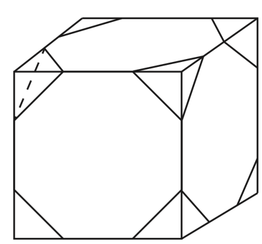
(A) 8/3 (B) e (C)$^\heartsuit$ 3 (D) π (E) the sum does not converge

Solution. The sum can be re-expressed as

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{2^i 3^j} = \left(\sum_{i=0}^{\infty}\frac{1}{2^i}\right)\left(\sum_{j=0}^{\infty}\frac{1}{3^j}\right) = \frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} = 2 \cdot \frac{3}{2} = 3.$$

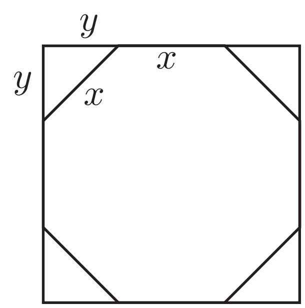
In 1737, Euler used the same idea to rewrite the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ as the product $\prod_p \frac{1}{1-1/p}$, where the product is over all primes p. Since the harmonic series diverges, Euler concluded that the sum of the reciprocals of the primes also diverges. In particular, there must be infinitely many primes!

Problem 17. Start with a 1 by 1 by 1 cube. Cut a tetrahedron off of each corner in such a way that the faces of the resulting polyhedron are all equilateral triangles or regular octagons. What is the edge length for each of those faces?



(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{2-\sqrt{2}}{2}$ (D)$^\heartsuit$ $\sqrt{2}-1$ (E) None of the above

Solution.



With x and y shown in the diagram, we need $y^2 + y^2 = x^2 = (1-2y)^2$. So $y = \frac{2\pm\sqrt{2}}{2}$. We discard the solution which is larger than 1. Then

$$x = 1 - 2y = 1 - 2\left(\frac{2-\sqrt{2}}{2}\right) = \sqrt{2} - 1.$$

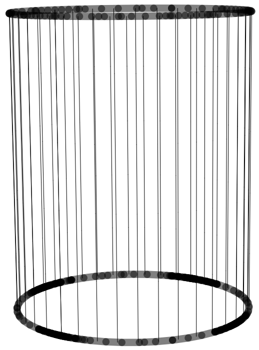
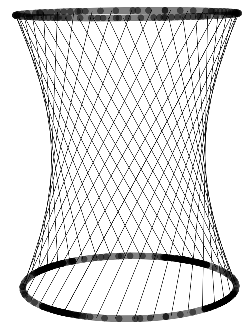
Problem 18. Two urns each contain the same number $N > 0$ of balls. Each ball is either red or black, and there are balls of both colors in each urn. From each urn we randomly choose a ball, note its color, return it to the urn, then choose and note again. Now suppose that the probability of choosing 2 red balls from the first urn equals the probability of choosing 2 red balls or 2 black balls from the second urn. What is the smallest N can be?

(A) 2 (B) 5 (C)$^\heartsuit$ 7 (D) 12 (E) it is not possible for those probabilities to be equal

Solution. Let r and b be the number of red and black balls, respectively, in the first urn, and let R and B be the corresponding number of balls in the second. Notice that $r + b = R + B = N$. The probability of choosing 2 red balls from the first urn is $(r/N)^2$, while the probability of drawing 2 red balls or 2 black balls from the second urn is $(R/N)^2 + (B/N)^2$. So we want $r^2 = R^2 + B^2$. The smallest nontrivial solution is $5^2 = 4^2 + 3^2$, so $N = R + B = 7$.

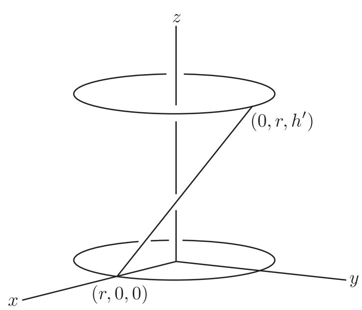
Notice that if we choose, with replacement, $n \geq 3$ balls, then we're looking for solutions to $r^n = R^n + B^n$; Fermat's last theorem says that there are no solutions for $n \geq 3$. This problem is often referred to as "Molina's urns".

Problem 19. Construct a cylinder as follows: Begin with two circles of radius r and join them by strings of length $h > 2r$ as shown. Now rotate the top circle 90° around the central axis of the cylinder. What is the radius of the "waist" of the resulting hyperboloid?

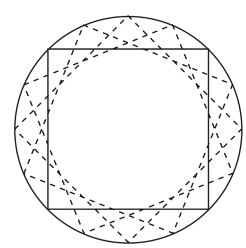
(A) r (B)$^\heartsuit$ $r/\sqrt{2}$ (C) $\sqrt{2}r$ (D) $\sqrt{\frac{1}{2}r^2+h^2}$ (E) $\sqrt{h^2-2r^2}$

Solution. Put coordinate axes in the cylinder as shown and consider the string joining $(r,0,0)$ to $(r,0,h)$.



After rotation, that string joins $(r,0,0)$ to $(0,r,h')$ for some height h'. The midpoint of the segment is on the waist, at $(r/2,r/2,h'/2)$, and the distance from this point to the central axis is $\sqrt{(r/2)^2+(r/2)^2}=r/\sqrt{2}$.

Alternative solution: Consider the view from the top. The central circle fits inside each of the squares shown. The squares have side length $s=\sqrt{2}r$, so the circle has radius $r/\sqrt{2}$.



Problem 20. Suppose $f(x)$ is a polynomial of degree 4 with integer coefficients and $f(2013)=f(2014)=1$. What is the largest number of integer solutions that $f(x)=0$ can have?

(A)$^\heartsuit$ 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution. Write $f(x)=a_0+a_1x+\cdots+a_4x^4$. Since $f(2014)=1$, we see that a_0 is odd, and so f has no even roots. On the other hand, if n is odd, then

$$\begin{aligned} f(n) &= a_0+a_1n+\cdots+a_4n^4 \equiv a_0+a_1+\cdots+a_4 \\ &\equiv a_0+a_1(2013)+a_2(2013)^2+a_3(2013)^3+a_4(2013)^4 = f(2013)\equiv 1 \pmod 2. \end{aligned}$$

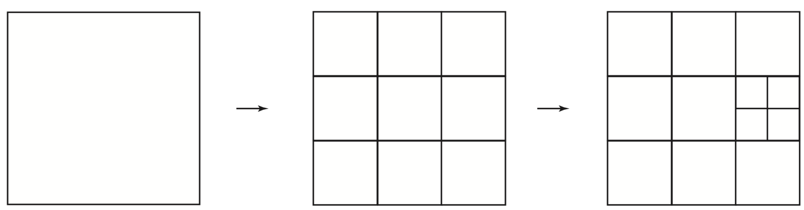
So f also has no odd roots.

3 Hard Problems

Problem 21. Begin with a square. You may, if you choose, partition it into squares by drawing lines parallel to the sides of the squares. All such partitioning lines must completely cross, but not extend beyond, the square being partitioned. You now have n^2 squares, for some natural number n. Notice that a square, once partitioned, is no longer included in the count.

You may now, if you choose, partition any of the squares in your partitioned square. And you may, if you choose, continue to partition squares for any finite number of steps.

Any such construction determines a certain number of squares, counting only squares which are not further subdivided. How many positive integers **do not** arise this way?



(A)$^\heartsuit$ 7 (B) 8 (C) 9 (D) 10 (E) infinitely many

Solution. First notice that, if you can construct n squares, then you can construct $n+3$ squares by subdividing one square into 4. Since we start with one square, we can use this observation repeatedly to construct any number of squares congruent to 1 (mod 3). By first constructing 9 squares, we can also construct any number of squares greater than 9 and congruent to 0 (mod 3). 2 is not a square mod 3, so it takes two steps to construct the smallest number of squares congruent to 2 modulo 3: $17 = 3^2 + 3^2 - 1$. So we can also construct any number greater than 17 and congruent to 2 (mod 3). This leaves 7 numbers: $2, 3, 5, 6, 8, 11, 14$. One can check directly that none of these numbers is constructible.

Problem 22. Let ℓ_1 be the line $y = x$ and ℓ_2 the line $y = 2x$. Let

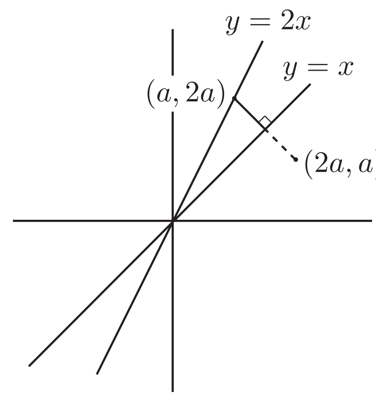
$$C = \{p \in \mathbb{R}^2 : d(p, \ell_1) + d(p, \ell_2) = 1\}.$$

Find the area of the region enclosed by C. **Note:** $d(p, \ell_m)$ is the perpendicular distance from the point p to the line ℓ_m.

(A) 10π (B) $2\sqrt{10}\pi$ (C) 1 (D)$^{\heartsuit}$ $\sqrt{40}$ (E) $\sqrt{20+6\sqrt{10}}$

Solution. Begin by finding the points on ℓ_2 1 unit away from ℓ_1: $(a, 2a)$ is on ℓ_2; its reflection over ℓ_1 is $(2a, a)$, so its distance to ℓ_1 is 1 if and only $d((a, 2a),(2a, a)) = \sqrt{2}|a| = 2$. So $\pm(\sqrt{2}, 2\sqrt{2})$ are on C.

Notice that those points are $\sqrt{10}$ units away from $(0, 0)$, so by symmetry, the points on ℓ_1 that are $\sqrt{10}$ units from $(0, 0)$ are also on C: $\pm(\sqrt{5}, \sqrt{5})$.

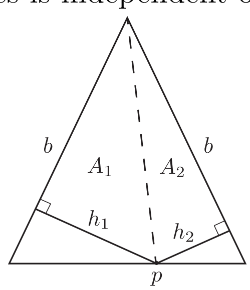


Claim: C is the **rectangle** with vertices at $\pm(\sqrt{5}, \sqrt{5})$, $\pm(\sqrt{2}, 2\sqrt{2})$.

Assuming this, the length of the rectangle is $d((\sqrt{5}, \sqrt{5}),(\sqrt{2}, 2\sqrt{2})) = \sqrt{20-6\sqrt{10}}$, and the width is $d((\sqrt{5}, \sqrt{5}),(-\sqrt{2}, -2\sqrt{2})) = \sqrt{20+6\sqrt{10}}$. So the area is

$$\sqrt{20-6\sqrt{10}} \cdot \sqrt{20+6\sqrt{10}} = \sqrt{40}.$$

We can see the claim using this very nice observation from plane geometry. If p is on the base of an isosceles triangle, then the sum of the distances from p to the two sides is independent of p.



Proof. Connect p to the opposite vertex and compare the areas of the regions formed to the area A of the whole triangle: $\frac{1}{2}bh = A = A_1 + A_2 = \frac{1}{2}bh_1 + \frac{1}{2}bh_2$, so $h_1 + h_2 = h$. $\square$

Alternative solution: Using the formula for the distance between a point and a line, we see that the set of points (x, y) belonging to C is the set of solutions to

$$\frac{|x-y|}{\sqrt{2}} + \frac{|2x-y|}{\sqrt{5}} = 1.$$

This becomes the tilted square $|X| + |Y| = 1$ under the change of variables

$$X = \frac{1}{\sqrt{2}}(x-y),$$

$$Y = \frac{1}{\sqrt{5}}(2x-y).$$

The map taking us from (x, y)-coordinates to (X, Y)-coordinates is a linear transformation of determinant $\frac{1}{\sqrt{10}}(1 \cdot (-1) - (-1) \cdot 2) = \frac{1}{\sqrt{10}}$. Since the sides of the tilted square have length $\sqrt{2}$, the tilted square itself has area 2, and so the area enclosed by C is $2 \cdot (1/\sqrt{10})^{-1} = \sqrt{40}$.

Problem 23. If $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 1 + \frac{a_n^2 - a_n a_{n-1} + 1}{a_n - a_{n-1}}$ for $n = 1, 2, 3, 4, \ldots$, find the integer closest to a_{10}.

(A) 4 (B) 11 (C) 15 (D)$^\heartsuit$ 16 (E) 24

Solution. The sequence beyond $a_6 = \frac{263}{30}$ is too messy to compute, so consider instead the sequence of first differences $b_{n+1} = a_{n+1} - a_n$, where $n \geq 0$. From this we'll be able to reconstruct $a_{10} = b_{10} + b_9 + \cdots + b_1$. Now b_n satisfies the recurrence $b_1 = 1$ and

$$\begin{aligned} b_{n+1} = a_{n+1} - a_n &= 1 + \frac{a_n^2 - a_n a_{n-1} + 1}{a_n - a_{n-1}} - a_n \\ &= 1 + \frac{1}{a_n - a_{n-1}} = 1 + \frac{1}{b_n}. \end{aligned}$$

Therefore, $b_n = \frac{f_{n+1}}{f_n}$, the ratio of consecutive Fibonacci numbers:

$$b_1, \ldots, b_{10} = 1,\ 2,\ \frac{3}{2},\ \frac{5}{3},\ \ldots,\ \frac{89}{55}.$$

Again, computing sums beyond about the 5th term is a bit messy, but we can take advantage of the fact that the b_i approximate the golden ratio $\phi \approx 1.618$ alternately from below and above. In particular,

$$1 + 2 + \frac{3}{2} + \frac{5}{3} + \cdots + \frac{89}{55} < 1 + 2 + \frac{3}{2} + 7 \cdot \frac{5}{3} = \frac{97}{6} = 16\ \frac{1}{6},$$

while

$$1 + 2 + \frac{3}{2} + \cdots + \frac{89}{55} > 1 + 2 + \frac{3}{2} + \frac{5}{3} + 6 \cdot \frac{8}{5} = 15\ \frac{23}{30}.$$

So the nearest integer is 16.

According to Wolfram Alpha, the actual value of a_{10} is $10796897/680680 = 15.8619\ldots$.

Problem 24. A classic, from Sam Loyd: Mary and Ann's ages add up to 44 years, and Mary is twice as old as Ann was when Mary was half as old as Ann will be when Ann is three times as old as Mary was when Mary was three times as old as Ann. How old is Ann?

(A) $13\ \frac{1}{2}$ (B) $14\ \frac{1}{2}$ (C) $15\ \frac{1}{2}$ (D)$^\heartsuit$ $16\ \frac{1}{2}$ (E) $17\ \frac{1}{2}$

Solution. Let M and A be Mary and Ann's ages. Then

$$M + A = 44.$$

"Mary is twice as old as Ann was..." is a reference to an earlier time, say B years earlier, so

$$M = 2(A - B).$$

What happened B years ago? "Mary was half as old as Ann will be..." This is comparing Mary's previous age to Ann's age at a future date, say C years later, so

$$M - B = \frac{1}{2}(A + C).$$

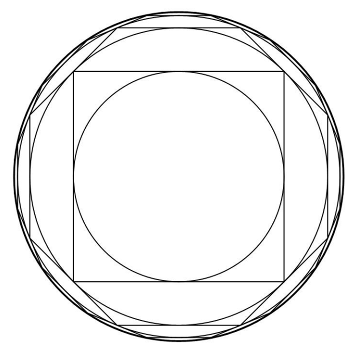
And what's happening C years later? "Ann is three times as old as Mary was..." Another reference to an earlier time, D years earlier, when "Mary was three times as old as Ann." So

$$\begin{aligned} A + C &= 3(M - D) \\ M - D &= 3(A - D). \end{aligned}$$

This is a system of 5 linear equations in 5 unknowns. Solve carefully and you'll find $A = 16\ \frac{1}{2}$.

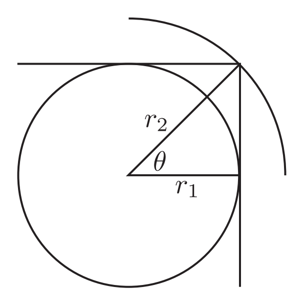
Problem 25. Let C_1 be a circle of radius 1, and P_1 a square circumscribed around C_1. For each $n \geq 2$, let C_n be the circle of radius r_n circumscribed around P_{n-1}, and let P_n be a regular 2^{n+1}-gon circumscribed around C_n. Find $\lim_{n\to\infty} r_n$.

(A) $\frac{1+\sqrt{5}}{2}$ (B) 2 (C) e (D) π (E)$^\heartsuit$ $\frac{\pi}{2}$



Solution. First, find r_2. In this diagram, $\theta = \pi/4$ and so $r_1/r_2 = \cos(\pi/4)$, i.e., $r_2 = r_1/\cos(\pi/4)$. In general, $r_{n-1}/r_n = \cos(\frac{\pi}{2^n})$ — the angle is halved each time the number of sides doubles. So

$$r_n = \frac{r_1}{\prod_{k=2}^{n} \cos(\pi/2^k)}.$$



What is $\prod_{k=2}^{n} \cos(\pi/2^k)$? Repeated use of the double-angle formula shows that

$$\begin{aligned} \sin(x) &= 2\sin(x/2)\cos(x/2) \\ &= 2^2 \sin(x/4)\cos(x/4)\cos(x/2) \\ &= \dots \\ &= 2^n \sin(x/2^n) \prod_{k=1}^{n} \cos(x/2^k). \end{aligned}$$

We can rewrite the coefficient $2^n \sin(x/2^n)$ as $\frac{x\sin(x/2^n)}{x/2^n}$. So we get

$$\frac{\sin(x)}{x} = \lim_{n\to\infty} \frac{\sin(x/2^n)}{x/2^n} \prod_{k=1}^{n} \cos(x/2^k) = \prod_{k=1}^{\infty} \cos(x/2^k).$$

Letting $x = \pi/2$, we get

$$\frac{2}{\pi} = \frac{\sin(\pi/2)}{\pi/2} = \prod_{k=1}^{\infty} \cos(\pi/2^{k+1}),$$

and so

$$\lim_{n\to\infty} r_n = \frac{r_1}{2/\pi} = \frac{\pi}{2}.$$

A special case of this analysis is the beautiful formula found by Francois Viète in 1593:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots .$$

Authors. Written by Mo Hendon and Paul Pollack. We thank Will Kazez for his help in drawing the pictures. The hyperboloid drawings in problem 19 are from the Wolfram Demonstrations Project. The figure in problem 25 is due to Daniel West.

Sources. Problem 7 is adapted from the 1986 American Invitational Mathematics Exam. Problem 11 is a variant of a 1983 contest problem from the New York City Interscholastic Mathematics League. Problem 18 is adapted from *Fifty Challenging Problems in Probability* by Frederick Mosteller. Problem 24 is due to Sam Loyd, a well-known puzzler of the late 19th century of somewhat dubious character; for more information on Loyd, see

`http://en.wikipedia.org/wiki/Sam_Loyd`