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WRITTEN TEST, 25 PROBLEMS / 90 MINUTES

October 20, 2012

WITH SOLUTIONS

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

1 Easy Problems

Problem 1. Calculate the sum

$$-1-2-3+4+5+6-7-8-9+10+11+12-\cdots+2008+2009+2010-2011-2012\,.$$

(A) -1017 (B)$^\heartsuit$ -1008 (C) -992 (D) -216 (E) None of the above

Solution. The sum as far as 2010 consists of 335 groups of six numbers each, each group totalling to $-(1+2+3)+(4+5+6)=9$. Thus, the sum is $9\cdot 335-(2011+2012) = 3015-4023=-1008$.

Problem 2. Frank owes \$5000 and Michael owes \$3000. If Frank had 2/3 of Michael's money in addition to his own, he could exactly pay all his debts; if Michael had 1/2 of Frank's money in addition to his own, he could pay all but \$100 of his debts. What is the total amount of money Frank and Michael have?

(A) \$4000 (B) \$4600 (C)$^\heartsuit$ \$5200 (D) \$5250 (E) None of the above

Solution. Let F be the amount of money Frank has (in dollars), and M the amount Michael has. We have

$$F+\frac{2}{3}M=5000 \qquad \text{and} \qquad \frac{1}{2}F+M=2900\,.$$

Solving, we find $M=600$ and $F=4600$, so $M+F=5200$.

Problem 3. A circle of radius 6 has its center on a circle of radius 5. How far apart are the two points of intersection?

(A) $3\sqrt{2}$ (B) $24/5$ (C) 5 (D) $5\sqrt{2}$ (E)$^\heartsuit$ $48/5$

Solution. We assume the smaller circle has its center at the origin. We may as well put the center of the larger circle at the point $(0, 5)$. We want the intersection points $(\pm x, y)$ of the circles $x^2 + y^2 = 25$ and $x^2 + (y-5)^2 = 36$. By algebra we find $y = \dfrac{7}{5}$, and so $x = \pm\dfrac{1}{5}\sqrt{625 - 49} = \pm\dfrac{24}{5}$. Thus, the distance between the two is $\dfrac{48}{5}$.

Problem 4. One pipe can fill a tank in 45 minutes and another can fill it in 30 minutes. If these two pipes are open and a third pipe is draining water from the tank, it takes 27 minutes to fill the tank. What is the time, in minutes, that it takes the third pipe alone to drain a full tank?

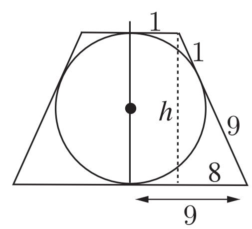
(A) 48 (B)$^\heartsuit$ 54 (C) 60 (D) $64\frac{1}{2}$ (E) None of the above

Solution. In one hour, pipe **1** fills $4/3$ of a tank and pipe **2** fills 2 tanks. Suppose pipe **3** empties x tanks in one hour. Since $27/60 = 9/20$, we have $\frac{9}{20}(\frac{4}{3} + 2 - x) = 1$, and so $x = 10/9$. That is, it takes pipe **3** $9/10$ hours, i.e., 54 minutes, to drain a full tank.

Problem 5. A sphere is inscribed in a truncated right circular cone (so that it is tangent at the top, the bottom, and along the lateral surface of the cone). If the radii at the top and bottom are 1 and 9, what is the radius of the sphere?

(A) $\sqrt{5}$ (B)$^\heartsuit$ 3 (C) $2\sqrt{5}$ (D) $9/2$ (E) 5

Solution. Recall that if a point X is external to a circle, the two line segments containing X tangent to the circle have equal length. We then obtain a right triangle with height h, base 8, and hypotenuse 10. So $h = 6$ and the radius of the sphere is 3.



Problem 6. Two real numbers are chosen at random between 0 and 10. What is the probability that their sum is greater than 8?

(A) 32% (B) 55% (C) 60% (D) 62.8% (E)$^\heartsuit$ 68%

Solution. We are considering the square $0 \le x, y \le 10$ in the xy-plane. It has area 100. We wish to discard the triangle given by $x + y \le 8$, which has area 32. This leaves us with an area of 68, which accounts for 68% of the possibilities.

Problem 7. Let A, B, and C be the real numbers defined by

$$A = \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}$$

$$B = \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}, \text{ and}$$

$$C = \sqrt{12 + \sqrt{12 + \sqrt{12 + \dots}}}.$$

Find $A + B + C$.

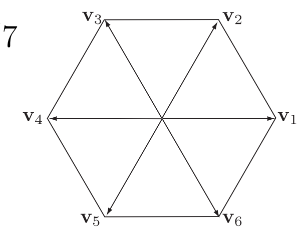
(A) 6 (B)$^\heartsuit$ 9 (C) 3π (D) $5+2\sqrt{5}$ (E) $7\sqrt{2}$

Solution. If $a>0$ and $x=\sqrt{a+\sqrt{a+\sqrt{a+\dots}}}$, then $x^2=a+x$, so $x=\frac{1}{2}\big(1+\sqrt{1+4a}\big)$. Thus,

$$\begin{aligned} A+B+C &= \tfrac{1}{2}\big((1+\sqrt{1+4\cdot 2})+(1+\sqrt{1+4\cdot 6})+(1+\sqrt{1+4\cdot 12})\big) \\ &= \frac{1}{2}(3+3+5+7)=9\,. \end{aligned}$$

Problem 8. A regular hexagon is inscribed in the unit circle and weights 1, 2, 3, 4, 5, 6 are placed *consecutively* at the vertices. How far from the center of the circle is the center of mass?

(A) 0 (B) 1/4 (C)$^\heartsuit$ 2/7 (D) $\sqrt{3}/4$ (E) 3/7



Solution. Put weight j at vertex $\mathbf{v}_j$, as pictured. Note that $\mathbf{v}_1+\mathbf{v}_3=\mathbf{v}_2$ and $\mathbf{v}_4+\mathbf{v}_6=\mathbf{v}_5$ because, e.g., $\cos 60^\circ = 1/2$. Then, since $\mathbf{v}_{j+3}=-\mathbf{v}_j$ for $j=1,2,3$,

$$\mathbf{v}_1+2\mathbf{v}_2+3\mathbf{v}_3+4\mathbf{v}_4+5\mathbf{v}_5+6\mathbf{v}_6=3(\mathbf{v}_4+\mathbf{v}_5+\mathbf{v}_6)=3(2\mathbf{v}_5)\,.$$

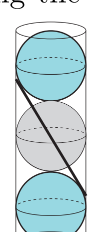
Since the total mass is $1+2+\cdots+6=21$, the center of mass is $(6/21)\mathbf{v}_5$, whose distance from the origin is 2/7.

Problem 9. Let $A=\{0,1,2,3,5,8,13,21,34,55\}$. Note that the nonzero numbers in A are consecutive Fibonacci numbers. Define $A+A+A=\{a+b+c : a,b,c\in A\}$ (note that a, b, c are not required to be distinct). What is the smallest positive integer that is not in $A+A+A$?

(A) 20 (B)$^\heartsuit$ 33 (C) 54 (D) 88 (E) 166

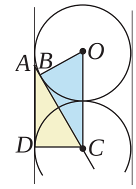
Solution. Every positive integer can be written uniquely as a sum of distinct Fibonacci numbers, provided no pair of consecutive numbers is used. (This can be proved by complete induction.) We want to find the smallest positive integer that cannot be written as the sum of three Fibonacci numbers: Take the sum of the four smallest non-consecutive Fibonacci numbers, $1+3+8+21=33$.

Problem 10. Instead of putting three tennis balls of radius 1 in a can, a mathematician's pencil (i.e., a line segment) is inserted in place of the middle ball in such a way that when it is tangent to both remaining balls (with both ends touching the can), the top ball is at its usual height. How long is the pencil?



(A) 3 (B) $2\sqrt{3}$ (C) 7/2 (D)$^\heartsuit$ 4 (E) $4\sqrt{3}$

Solution. As we see in the diagram, the key is congruent triangles. Half the pencil is AC, and $\triangle ADC \cong \triangle CBO$. The triangles are clearly similar (e.g., $\angle OCB + \angle ACD = 90°$), but since OB and CD are both radii, they are equal; thus, the triangles are congruent. Therefore, $AC = OC = 2$, and the pencil has length 4.



2 Medium Problems

Problem 11. How many pairs of rational numbers (a, b) are there for which

$$(a + bi)^7 = a - bi\,?$$

(Remember that $i^2 = -1$.)

(A) 2 (B) 4 (C)$^\heartsuit$ 5 (D) 8 (E) 9

Solution. Let $z = a + bi$. We have $|z|^7 = |\overline{z}| = |z|$, so either $z = 0$ or $|z| = 1$. In the latter case, we have $\overline{z} = 1/z$, so $z^8 = 1$. We ask how many solutions have rational coordinates. The eighth roots of unity are ± 1, $\pm\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$, and $\pm i$. Four of these have rational coordinates. This makes five solutions, in total.

Problem 12. Suppose θ and ϕ are real numbers for which

$$\sin(\theta) + \sin(\phi) = 1/2 \qquad \text{and} \qquad \cos(\theta) + \cos(\phi) = -1/2\,.$$

What is the value of $\sin(\theta + \phi)$?

(A)$^\heartsuit$ -1 (B) $-1/2$ (C) 0 (D) $\sqrt{3}/2$ (E) 1

Solution. Adding the equations, we get

$$\sin(\theta) + \cos(\theta) + \sin(\phi) + \cos(\phi) = 0\,.$$

Multiplying by $1/\sqrt{2}$ and recalling the addition formula $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$, we have

$$\sin\left(\theta + \frac{\pi}{4}\right) + \sin\left(\phi + \frac{\pi}{4}\right) = 0\,.$$

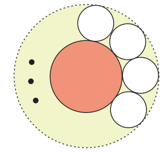
It follows that either

$$\begin{aligned} &\theta + \pi/4 = -(\phi + \pi/4)\,, &&\text{from which we obtain} \quad \theta = -\phi - \pi/2\,, \quad \text{or} \\ &\theta + \pi/4 = \pi + \phi + \pi/4\,, &&\text{from which we obtain} \quad \theta = \pi + \phi\,. \end{aligned}$$

The latter choice contradicts the first of the given equations. Thus, $\theta + \phi = -\pi/2$, and $\sin(\theta + \phi) = -1$.

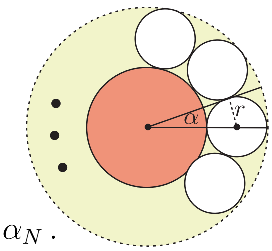
Alternative solution: Consider the unit complex numbers $z = \cos\theta + i\sin\theta$ and $w = \cos\phi + i\sin\phi$. Then we are given the fact that $z + w = \frac{1}{2}(-1 + i) = \frac{1}{\sqrt{2}}\zeta$, where $\zeta = \cos(3\pi/4) + i\sin(3\pi/4)$. Multiplying by $1/\zeta = \overline{\zeta}$ rotates the picture an angle $-3\pi/4$. In particular, if we let $z' = \overline{\zeta}z$ and $w' = \overline{\zeta}w$, then $z' + w' = 1/\sqrt{2}$, and so it follows that $w' = \overline{z'}$ and $z'w' = 1$. From this we have $zw = (\zeta z)(\zeta w) = \zeta^2 = -i$. But $zw = \cos(\theta + \phi) + i\sin(\theta + \phi)$, so $\sin(\theta + \phi) = -1$.

Problem 13. N congruent circles are packed tightly around a circle of radius 1 and inside a concentric circle (which shrinks as N gets bigger). As N goes to infinity, what fraction of the area of the outer ring is covered by the circles?



(A) $\pi/6$ (B) $3/4$ (C)$^{\heartsuit}$ $\pi/4$ (D) $3\pi/10$ (E) 1

Solution. If we pack N circles of radius r around the unit circle, each of them subtends an angle of $2\alpha = 2\pi/N$ at the origin. Then $\alpha = \alpha_N = \pi/N$ and $\sin\alpha_N = r/(1+r)$. The total area of these circles is $N\pi r^2$ and the fraction of the area of the outer ring is



$$\frac{N\pi r^2}{\pi\big((1+2r)^2-1\big)} = \frac{Nr^2}{4(r+r^2)} = \frac{N}{4}\frac{r}{1+r} = \frac{N}{4}\sin\alpha_N\,.$$

We know that $N\sin\alpha_N = N\sin\dfrac{\pi}{N} \to \pi$ as $N \to \infty$. (Without calculus, we can recognize this as follows: The circumference of an inscribed regular N-gon in the unit circle is $2N\sin(\pi/N)$, and this approaches 2π as $N\to\infty$.) Therefore, our limiting fraction is $\pi/4$.

Problem 14. For each number $n > 1$, let

$$s_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots\,.$$

What is the value of the infinite sum $s_2 + s_3 + s_4 + \dots$?

(A) 2 (B) $\pi^2/6$ (C) e^π (D) infinite (E)$^{\heartsuit}$ None of the above

Solution. We have

$$\begin{aligned}\sum_{n=2}^{\infty} s_n &= \sum_{n=2}^{\infty}\sum_{j=2}^{\infty}\frac{1}{j^n} = \sum_{j=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{j^n}\\ &= \sum_{j=2}^{\infty}\frac{\frac{1}{j^2}}{1-\frac{1}{j}} = \sum_{j=2}^{\infty}\frac{1}{j(j-1)} = \sum_{j=2}^{\infty}\left(\frac{1}{j-1}-\frac{1}{j}\right) = 1\,.\end{aligned}$$

Problem 15. How many subsets of $\{1, 2, 3, 4, \dots, 10\}$ contain no pair of consecutive integers?

(A) 105 (B)$^{\heartsuit}$ 144 (C) 256 (D) 512 (E) None of the above

Solution. Let A_n be the number of subsets of $\{1, 2, \dots, n\}$ without a pair of consecutive integers. We claim that $A_n = F_{n+2}$, where F_m denotes the m^{th} Fibonacci number. The asserted equality is obvious for $n = 0$ and $n = 1$. To prove it in general, it is enough to demonstrate the recurrence relation $A_{n+1} = A_n + A_{n-1}$.

Given a subset of $\{1, 2, \dots, n\}$ with no consecutive integers, let us count how many ways there are to extend it to a subset of $\{1, 2, \dots, n+1\}$ with no consecutive integers. (By an extension of a given set, we mean a subset of $\{1, 2, \ldots, n+1\}$ whose intersection with $\{1, 2, \ldots, , n\}$ is the original set.) If our given subset S contains n, then S itself is the only extension of $\{1, 2, \ldots, n+1\}$. If S does not contain n, then there are two extensions, namely S and $S \cup \{n+2\}$. So

$$\begin{aligned} A_{n+1} &= \#\{S \text{ containing } n\} + 2\#\{S \text{ not containing n}\} \\ &= (\#\{S \text{ containing } n\} + \#\{S \text{ not containing } n\}) + \#\{S \text{ not containing } n\} \\ &= A_n + A_{n-1}, \end{aligned}$$

as desired.

In the original problem, $n = 10$, and so the answer is $F_{12} = 144$.

Problem 16. The future UGA math department has offices numbered 1 through 2012. A madman breaks into the building and scrawls a 1 on the markerboard outside each office. He then turns around and writes a 2 on the markerboard of each *even-numbered* office. Turning around again, he puts a 3 on those whiteboards whose office numbers are multiples of 3. He continues the same process for 2009 more steps. Finally, he leaves the building and turns himself in to the police.

When the madman is gone, FBI consultant Charlie Eppes totals the numbers on each whiteboard. On how many whiteboards does Charlie find an even sum?

(A) 44 (B) 1024 (C)$^\heartsuit$ 1937 (D) 1998 (E) None of the above

Solution. We begin by counting the number of whiteboards for which the sum is *odd*. After the dust has settled, the sum of the numbers on the n^{th} whiteboard is precisely the sum of the positive divisors of n, usually denoted $\sigma(n)$. If we write the prime factorization of n in the form

$$n = p_1^{e_1} \cdots p_k^{e_k},$$

then it is not hard to prove that

$$\sigma(n) = \prod_{i=1}^{k} (1 + p_i + p_i^2 + \cdots + p_i^{e_i}).$$

In order for the right-hand product to be odd, it is necessary and sufficient that each of the factors be odd. Now the sum $1 + p_i + p_i^2 + \cdots + p_i^{e_i}$ is odd if and only if $p_i = 2$ or e_i is even. It follows that $\sigma(n)$ is odd precisely when we can write $n = m^2$ or $n = 2m^2$ for some integer m.

The number of squares up to 2012 is $\left[\sqrt{2012}\right] = 44$, and the number of integers which are twice a square up to 2012 is $\left[\sqrt{1006}\right] = 31$. So the number of whiteboards with an odd sum is 75, and the number with an even sum is $2012 - 75 = 1937$.

Problem 17. A regular hexagon is inscribed in the unit circle and weights 1, 2, 3, 4, 5, 6 are placed in *some* order at the vertices. What is the maximum distance from the center of the circle that you can arrange the center of mass to be?

(A) $2/7$ (B) $1/3$ (C) $4\sqrt{3}/21$ (D)$^\heartsuit$ $2\sqrt{13}/21$ (E) $\sqrt{57}/21$

Solution. Following the solution of problem #8, we want a permutation σ of $\{1,2,\dots,6\}$ so that $\mathbf{S} = \sum_{j=1}^{6} \sigma(j)\mathbf{v}_j$ will have maximum length. (Then we multiply this vector by $1/21$ to obtain the center of mass.) Recalling that $\mathbf{v}_{j+3} = -\mathbf{v}_j$, $j = 1,2,3$, we rewrite this sum as $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$, where $a = \sigma(1) - \sigma(4)$, $b = \sigma(2) - \sigma(5)$, and $c = \sigma(3) - \sigma(6)$. For the sum to be as large a vector as possible, we want three consecutive vertices to have positive coefficients, and we may as well take these to be $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Since the coefficients add to 21, the largest $a+b+c$ can be is $(4+5+6) - (1+2+3) = 9$. Now consider

$$\mathbf{S} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} + c\begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2a+b-c \\ \sqrt{3}(b+c) \end{bmatrix}.$$

Then

$$\begin{aligned} \|\mathbf{S}\|^2 &= \frac{1}{4}\big((2a+b-c)^2 + 3(b+c)^2\big) = (a^2+b^2+c^2) + (ab-ac+bc) \\ &= (a+b+c)^2 - (ab+bc+3ac) \end{aligned}$$

We assert that the way to maximize this quantity is to make $a+b+c = 9$ and *minimize* $ab+bc+3ac$. We tabulate the possibilities (using the a-c symmetry), remembering that $1 \le a,b,c \le 5$:

a	b	c	$b(a+c)+3ac$	$\|\mathbf{S}\|^2$
3	3	3	45	36
2	4	3	38	43
2	5	2	32	49
1	**5**	**3**	**29**	**52**
1	4	4	32	49
1	3	5	33	48

Corresponding to the optimal $\mathbf{S}$ is the center of mass at distance $2\sqrt{13}/21 \approx 0.34$ from the origin. (This shows, by the way, that considering the next case, where $a+b+c = 7$, cannot lead to an optimal result.)

Problem 18. How many times do you expect to have to roll a fair die in order to get each of the numbers one through six to appear?

(A) 6 (B) 10 (C) 12 (D)$^\heartsuit$ 15 (E) None of the above

Solution. The heuristic is this: If the probability of a particular outcome of an experiment is p, then in N repetitions, we would expect on average to have that outcome appear pN times (at least when N is very large). So in the long run, the average number of repetitions we must have for each successful outcome is $N/Np = 1/p$. For example, we expect, on average, to toss a coin *twice* in order to get "heads."

It's easy now to solve the problem. With the first roll you get the first number (say 1). The probability of getting any one of numbers 2 through 6 in a roll is 5/6, so we expect to have to roll 6/5 times to have this result. Having obtained one of these (say 2), the probability of getting one a roll of 3 through 6 is 4/6, so we expect to have to make 6/4 rolls of the die. Having obtained one of these (say 3), the probability getting one of cards 4 through 6 is 3/6, and so we expect to make 6/3 rolls to do so. To get one of the remaining values we expect to make 6/2 rolls, and, last, when we lack one value, we expect to make 6 rolls in order to obtain it. So, in sum, we expect to have to make

$$1+\frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+6=6\left(\frac{1}{6}+\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1\right)=\frac{147}{10}=14.7$$

rolls. Thus, we expect to need 15 rolls of the die.

Problem 19. Let d_n be the final digit before the decimal point of $(2+\sqrt{5})^n$. For example, $d_3=6$, since $(2+\sqrt{5})^3=76.0131\ldots$. Find $d_1+d_2+d_3+\cdots+d_{2012}$.

(A)$^\heartsuit$ 9054 (B) 9060 (C) 10060 (D) 11066 (E) None of the above

Solution. Let $a_n=[(2+\sqrt{5})^n]$, and notice that d_n is just the units digit of a_n. Set

$$b_n=(2+\sqrt{5})^n+(2-\sqrt{5})^n\,.$$

Then b_n is an integer; moreover, the summand $(2-\sqrt{5})^n$ is smaller than 1 in absolute value and alternates in sign. It follows that $a_n=b_n-1$ if n is even, and $a_n=b_n$ if n is odd. Also, the b_n satisfy the recurrence

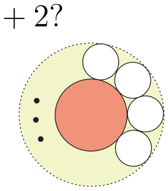
$$b_{n+1}=4b_n+b_{n-1}.$$

(For example, this is easy to verify by induction.) Now starting with $n=0$, the sequence of units digits of the b_n is $2,4,8,6,2,4,8,6,\ldots$. So the sequence of units digits of the a_n is (again starting with $n=0$) $1,4,7,6,1,4,7,6,\ldots$. Since 2012 is a multiple of 4, the sum of the first 2012 values d_n is

$$(1+4+7+6)\cdot\frac{2012}{4}=9054.$$

Problem 20. For how many positive integers n can we fit tightly packed congruent circles of radius 1 in the ring between concentric circles of radii n and $n+2$?

(A) 0 (B)$^\heartsuit$ 1 (C) 2 (D) 5 (E) infinitely many



Solution. Obviously (as we know from the ciphering round), we can pack 6 unit circles between concentric circles of radii 1 and 3. But this is the *only* solution! As in the solution of problem #13, if we can pack N unit circles between concentric circles of radii n and $n+2$ if and only if $\sin(\pi/N) = 1/(n+1)$. So we want to know for how many values of $\theta = \pi/N$ we can have $\sin\theta = 1/(n+1)$ for some $n \in \mathbb{N}$. Equivalently, by Euler's formula, we are asking that $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$ be roots of the polynomial $z^N = e^{i\pi} = -1$. This means that $e^{\pm i\theta}$ is what's called an *algebraic integer* (a root of a polynomial with *integer* coefficients and leading coefficient 1).

Now, here's an important (but not totally obvious) fact: The set of algebraic integers forms a subring of $\mathbb{C}$; that is, the set is closed under addition and multiplication. Thus, if $e^{i\theta}$ is an algebraic integer, then so is $2\sin\theta = i(e^{i\theta} - e^{-i\theta})$. But the only *rational numbers* that are algebraic integers are integers themselves, so we must have $2\sin\theta = 0$, ± 1, or ± 2, so $n+1 = 1$ or 2. These correspond to $N = 2$ and $N = 6$, respectively. The case $N = 2$ cannot occur in our situation.

Problem 21. An integer $n \geq 0$ is called *automorphic* if the decimal expansion of n^2 ends in the same digits as n (in order). For example, 0 is automorphic, since $0^2 = 0$, and 76 is automorphic, since $76^2 = 5776$. How many automorphic numbers are there between 0 and $10,000$, inclusive?

(A) 4 (B) 6 (C) 8 (D)$^{\heartsuit}$ 9 (E) 10

Solution. The positive integer n is a d-digit "automorph" precisely when n is a solution of the congruence $n^2 \equiv n \pmod{10^d}$. Equivalently, n must be a solution of the simultaneous congruences

$$n^2 \equiv n \pmod{5^d} \quad \text{and} \quad n^2 \equiv n \pmod{2^d}.$$

The first of these congruences translates to the requirement that 5^d divides $n^2 - n = n(n-1)$. Since n and $n-1$ cannot both be multiples of 5, the divisibility holds exactly when 5^d divides n or 5^d divides $n-1$. Similarly, the second congruence displayed above holds exactly when 2^d divides n or 2^d divides $n-1$. We conclude that a d-digit number n is an automorph exactly when

$$n \equiv 0, 1 \pmod{5^d} \quad \text{and} \quad n \equiv 0, 1 \pmod{2^d}.$$

These simultaneous congruences always have two solutions (mod 10^d), 0 and 1; for $d = 1$ we also obtain the two remaining single-digit automorphs, 5 and 6. If $d > 1$, the only possibilities for automorphs are solutions to

$$n \equiv 0 \pmod{5^d} \quad \text{and} \quad n \equiv 1 \pmod{2^d}$$

or

$$n \equiv 1 \pmod{5^d} \quad \text{and} \quad n \equiv 0 \pmod{2^d}.$$

For small d, the moduli are small enough that these systems can be solved by inspection (cycling through solutions to the first congruence until one reaches a solution of the second). For $d = 2$, the solutions are $n = 25$ and $n = 76$, which give us the two-digit automorphs. For $d = 3$, the solutions are $n = 625$ and $n = 376$, which are the 3-digit automorphs. When $d = 4$, the solutions are $n = 625$ and $n = 9376$; only the latter is a 4-digit automorph. So the set of automorphs up to 10000 is $\{0, 1, 5, 6, 25, 76, 376, 625, 9376\}$. So there are 9 such numbers.

3 Hard Problems

Problem 22. For $1 \le m < n$ consider the equation

$$1 + 2 + \cdots + m = (m+1) + (m+2) + \cdots + n\,.$$

There are two "small" solutions, namely $(m,n) = (2,3)$ and $(m,n) = (14,20)$. How many solutions are there with $20 < n \le 2012$?

(A) 0 (B) 1 (C)$^{\heartsuit}$ 2 (D) 3 (E) 4

Solution. The given equation is equivalent to $2\sum_{i=1}^{m} i = \sum_{i=1}^{n} i$, so we need to solve $m(m+1) = n(n+1)/2$. Completing the square, we get

$$\begin{aligned} \left(m+\frac{1}{2}\right)^2 - \frac{1}{4} &= \frac{1}{2}\left(\left(n+\frac{1}{2}\right)^2 - \frac{1}{4}\right)\,, \quad \text{and so} \\ 2(2m+1)^2 &= (2n+1)^2 + 1\,. \end{aligned}$$

That is, $(2m{+}1, 2n{+}1)$ is an *odd* solution of Pell's equation $2x^2 = y^2{+}1$. Now, (x,y) is a solution of Pell's equation if and only if y/x is a continued fraction approximation of $\sqrt{2}$. (See, for example, `http://mathworld.wolfram.com/PellEquation.html`.) Since $\sqrt{2} = [1; 2, 2, 2, 2, \ldots]$, i.e.,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}\,,$$

its convergents are

$$1,\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70},\ \frac{239}{169},\ \frac{577}{408},\ \frac{1393}{985},\ \frac{3363}{2378},\ \frac{8119}{5741},\ \ldots\,.$$

The convergents with odd numerators and denominators correspond to solutions (m,n) as follows:

convergent	(m,n)
1	$(0,0)$
7/5	$(2,3)$
41/29	$(14,20)$
239/169	$(84,119)$
1395/985	$(492,697)$

Any remaining convergents correspond to $n > 2012$, and so there are two solutions with $20 < n \le 2012$.

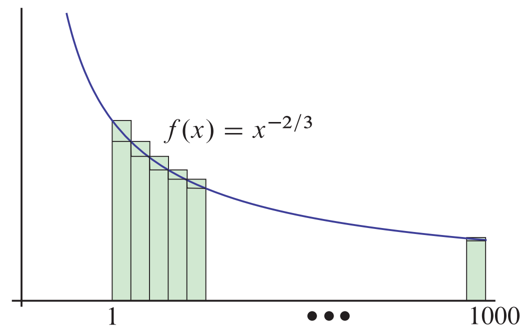
Problem 23. Recall that $[x]$ denotes the greatest integer $\leq x$. Determine

$$\left[\frac{1}{1^{2/3}} + \frac{1}{2^{2/3}} + \cdots + \frac{1}{1000^{2/3}}\right].$$

(A) 22 (B) 26 (C) 28 (D) 499 (E)$^\heartsuit$ None of the above

Solution. As we see from the picture,

$$\sum_{n=1}^{1000} n^{-2/3} = 1 + \sum_{n=2}^{1000} n^{-2/3} < 1 + \int_1^{1000} t^{-2/3}\,dt = 28\,,$$



and

$$\sum_{n=1}^{1000} n^{-2/3} > \int_1^{1000} t^{-2/3}\,dt = 27\,.$$

So the integer part is 27. (This is the integral test from the theory of infinite series.)

Alternative solution: Expanding out directly, it is easy to prove that for each $n \geq 1$,

$$\left(1 + \frac{1}{3n}\right)^3 > 1 + 1/n\,, \quad \text{whereas} \quad \left(1 - \frac{1}{3n}\right)^3 > 1 - 1/n.$$

After a bit of rearranging, we get

$$3((1 + 1/n)^{1/3} - 1) < \frac{1}{n} < 3(1 - (1 - 1/n)^{1/3}),$$

so that, multiplying through by $n^{1/3}$,

$$3((n+1)^{1/3} - n^{1/3}) < \frac{1}{n^{2/3}} < 3(n^{1/3} - (n-1)^{1/3}).$$

Now sum the right-hand inequality from $n = 2$ to $n = 10^3$, and add $1 = \frac{1}{1^{2/3}}$; this gives that the sum inside the floor function is smaller than 28. On the other hand, summing the left-hand inequality from $n = 1$ to $n = 10^3$ gives that the sum inside the floor function is $> 3((1001)^{1/3} - 1) > 3(10 - 1) = 27$. So again, we find that the integer part of the sum is 27.

Problem 24. The Euler function $\phi(n)$ is defined to be the number of integers between 1 and n (inclusive) which do not share any common factor > 1 with n. For example, $\phi(10) = 4$, since $1, 3, 7$, and 9 is the full list of the numbers in $\{1, 2, 3, \ldots, 10\}$ without a common factor with 10. For how many integers $1 \leq n \leq 2012$ is $\phi(n) = n/3$?

(A) 4 (B)$^\heartsuit$ 30 (C) 60 (D) 335 (E) None of the above

Solution. By inclusion-exclusion, one can prove that for each positive integer n,

$$\frac{\phi(n)}{n} = \prod_{p|n}(1 - 1/p), \tag{1}$$

where the right-hand product is over the primes dividing n. From this, we deduce the following two useful facts:

- The ratio $\phi(n)/n$ depends only on the set of primes dividing n (and not the powers to which they occur).
- The largest prime p dividing the denominator of $\phi(n)/n$ is the same as the largest prime dividing n. (This is because p cannot divide a product of $p' - 1$'s, for smaller primes p' dividing n.)

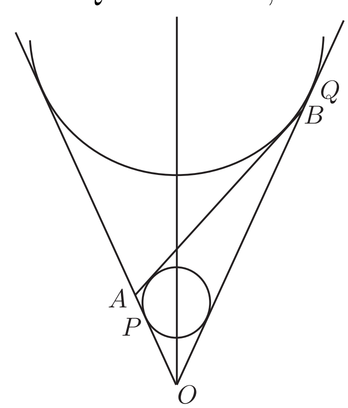
It follows that if $\phi(n)/n = 1/3$, then 3 is the largest element of the set of primes S dividing n. So the only possibilities for S are $\{2, 3\}$ and $S = \{3\}$. Since $\phi(3)/3 = 1/2$, the only genuine possibility is $S = \{2, 3\}$.

So the integers n with $\phi(n)/n = 1/3$ are exactly the integers of the form 2^a3^b, with $a, b \geq 1$. For each b, let us count the number of possible a so that $2^a3^b \leq 2012$. Since $3^7 = 2187 > 2012$, we need $1 \leq b \leq 6$. If $b = 1$, we need $1 \leq a \leq 9$. If $b = 2$, we need $1 \leq a \leq 7$. If $b = 3$, we need $1 \leq a \leq 6$. If $b = 4$, we need $1 \leq a \leq 4$. If $b = 5$, we need $1 \leq a \leq 3$. Finally, if $b = 6$, only $a = 1$ works.

So the total number of such n up to 2012 is

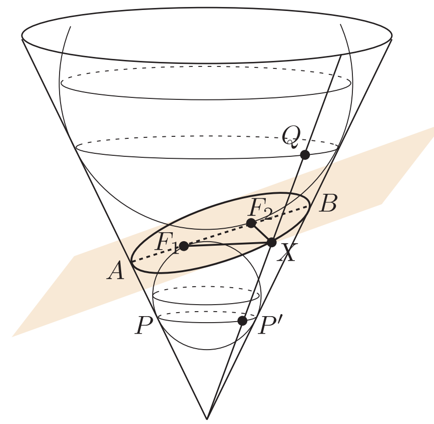
$$9 + 7 + 6 + 4 + 3 + 1 = 30.$$

Problem 25. Two circles, of radii 1 and 3, respectively, are inscribed in $\angle POQ$, as shown. They are also tangent to $\overline{AB}$, with P on $\overline{OA}$ and B on $\overline{OQ}$. If $OP = 3$, what is AB?

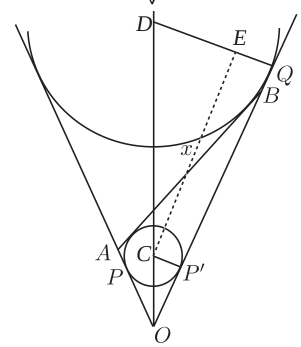


(A) 4 (B) 5 (C) $4\sqrt{3}$ (D)$^\heartsuit$ 6 (E) $6\sqrt{3}$

Solution. We do not know any solution using just plane geometry. In 1822 Dandelin gave the following proof that slicing a cone with a plane (at the appropriate angles) gives an ellipse. As pictured to the right, if we slice with a plane, there are two inscribed spheres tangent to the plane (one below, one above). The points of tangency (F_1 and F_2 in the picture) are the foci of the ellipse. Recall that if a point X is external to a circle, the two line segments containing X tangent to the circle have equal length. So we have $XF_1 = XP'$ and $XF_2 = XQ$. But $XP' + XQ = P'Q$ is constant, independent of which generator of the cone we're on. So we see that the ellipse is the locus of points X the sum of whose distances from F_1 and F_2 is constant. But when X is in position A or B, that distance is AB. Thus, $AB = P'Q$.



Now we revert to plane geometry. Let C be the center of the circle of radius 1 and D the center of the circle of radius 3. Let E be the point on $\overline{DQ}$ so that $\overline{CE}$ is parallel to $\overline{P'Q}$. Then, given that $OP = OP' = 3$, we have $\dfrac{1}{3} = \dfrac{2}{x}$, so $x = P'Q = 6$. This means, finally, that $AB = 6$!



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