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Written test, 25 problems / 90 minutes
October 17, 2009

WITH SOLUTIONS

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

1 Easy Problems

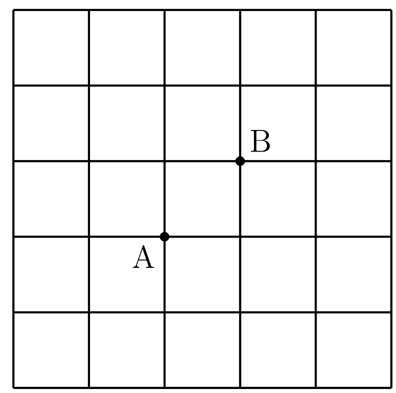
Problem 1. John has a bunch of shoes in his closet, but only 2/3 of the left shoes have matching right shoes, and only 3/5 of the right shoes have matching left shoes. What fraction of the shoes are parts of matching pairs? (No shoe is part of two pairs.)

(A)$^\heartsuit$ 12/19 (B) 19/30 (C) 21/30 (D) 12/15 (E) None of the above

Solution. Let us say there are N pairs, so that there are N left and N right shoes matched. 2/3 of all left shoes are matched, so there are $(3/2)N$ left shoes. 3/5 of all right shoes are matched, so there are $(5/3)N$ right shoes. Altogether, there are $(3/2)N + (5/3)N = (19/6)N$ shoes, and of these $2N$ are matched. So the portion is

$$\frac{2N}{(19/6)N} = \frac{12}{19}$$

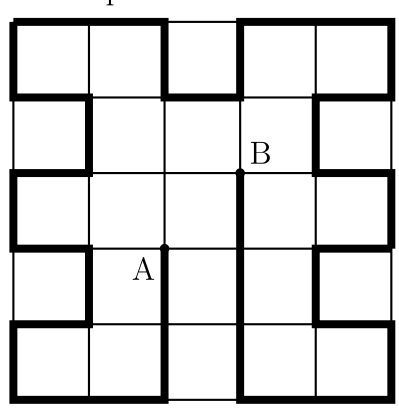
Problem 2. A curious tourist wants to go for a walk on the streets of the Old Town from his hotel (the point A on the map below) to the train station (the point B) using the longest way possible but never passing through the same point twice. (He can only move on the grid.)



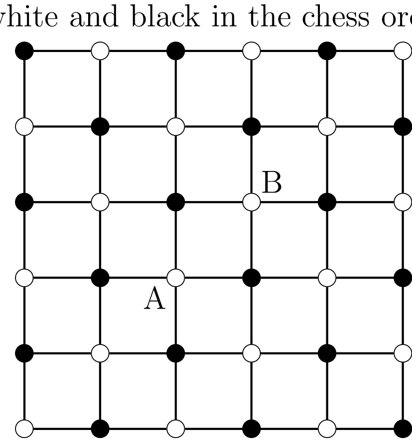
If we consider any interval of length 1 to be a street, how many streets can the tourist traverse?

(A) 33 (B)♡ 34 (C) 35 (D) 36 (E) None of the above

Solution. One possible solution with 34 streets is shown on the picture below. Let us show that a larger number is impossible.



Altogether in the old town there are 36 intersections. Every time the tourist passes a street, he comes to a new intersection. Therefore, he can not go through more than 35 streets. Let us show that he can not go through 35 streets either. For this, let us color the intersections in white and black in the chess order (see the picture below).



Every time he traverses a street, the tourist comes to an intersection of the opposite color. The hotel and the train station have the same color. Therefore, any route will have an even number of streets.

Problem 3. Every morning Joe walks for 1 mile along the tram tracks and counts the trams passing him from behind and coming towards him. During the year, he counted 100 of the former and 300 of the latter. Joe's speed is 3 mph. What is the tram's speed, in mph?

(A) 5 (B) 7 (C) 9 (D) 10 (E)$^\heartsuit$ None of the above

Solution. Let v be the tram's speed. The numbers 100 and 300 are proportional to the speeds of the trams relative to Joe. Therefore,

$$\frac{v+3}{v-3} = \frac{300}{100} = \frac{3}{1}$$

and $v = 6$.

Problem 4. If a child is born in 2009, what will be the next year that both her age and the year are perfect squares?

(A) In 4 or 9 years (B)$^\heartsuit$ In 16 or 25 years (C) In 36 or 49 years (D) A different answer (E) Never

Solution. The next perfect square larger than 2009 is $2025 = 45^2$. Luckily, this works: $2025 = 2009 + 16$.

Problem 5. Consider the number

$$15! = 1 \cdot 2 \cdot 3 \cdots 14 \cdot 15$$

Add its digits to obtain a new number. Add its digits to obtain a new number, and continue this process until you get a single digit. What is it?

(A) 0 or 1 (B) 2 or 3 (C) 4 or 5 (D) 6 or 7 (E)$^\heartsuit$ 8 or 9

Solution. An integer and the sum of its digits have the same remainder modulo 9 because $10^k = 1 \pmod 9$ for any k. Now, 15! is divisible by 9, so in the end we will get either 0 or 9. We can not get 0 if we don't start with 0. So the answer is 9.

Problem 6. Two positive real numbers have an average of 10. Which of the following must be true about μ, the average of their reciprocals?

(A) $\mu = 10$ (B) $\mu = 1/10$ (C) μ can be any real number (D)$^\heartsuit$ $\mu \geq \dfrac{1}{10}$ (E) $\mu \leq \dfrac{1}{10}$

Solution. If the numbers are x and y, then

$$\mu = \frac{\frac{1}{x} + \frac{1}{y}}{2} = \frac{\frac{x+y}{2}}{xy} = \frac{10}{xy}\,.$$

Since the *largest* the product can be is 100, obtained when $x = y = 10$ (because $xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 \le \left(\frac{x+y}{2}\right)^2 = 10^2$), we see that $\mu \ge \dfrac{10}{100} = \dfrac{1}{10}$.

This problem demonstrates the *arithmetic mean-harmonic mean* inequality, that $\frac{x+y}{2} \ge \frac{2}{\frac{1}{x}+\frac{1}{y}}$. This may be generalized: for any positive real numbers, $a_1, a_2, \ldots, a_n$,

$$\begin{aligned}\max(a_1, a_2, \ldots, a_n) &\ge \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \ge \frac{a_1 + a_2 + \cdots + a_n}{n} \\ &\ge \sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \\ &\ge \min(a_1, a_2, \ldots, a_n).\end{aligned}$$

Furthermore, each equality holds if and only if the a_i are equal, that is $a_1 = a_2 = \cdots = a_n$. At first, it may seem that these inequalities are difficult to remember, but they are special cases of the *generalized mean*:

$$m(t) \equiv \left(\frac{1}{n}\sum_{k=1}^{n} a_k^t\right)^{1/t}.$$

Then, the quadratic mean (root mean square), arithmetic mean, and harmonic mean are obviously $m(2)$, $m(1)$, and $m(-1)$ respectively. Less obviously, the maximum $\max(a_1, a_2, \ldots, a_n)$ is actually $m(\infty) = \lim_{t\to\infty} m(t)$ and $\min(a_1, a_2, \ldots, a_n)$ is $m(-\infty)$. But most interestingly, the geometric mean, which does not particularly resemble a mean of the form $m(t)$ is actually $m(0) = \lim_{t\to 0} m(t)$.

In terms of the $m(t)$, the long chain of inequalities above can be rewritten more simply as

$$m(\infty) \ge m(2) \ge m(1) \ge m(0) \ge m(-1) \ge m(-\infty).$$

This is interesting because $\infty > 2 > 1 > 0 > -1 > -\infty$. In fact, this is an excellent way to remember the long chain of inequalities, and it all works because of the generalized mean inequality: If $k > l$, then $m(k) \ge m(l)$ with equality if and only if the a_i are equal. Here we see that it makes sense for $\max = m(\infty)$, $\min = m(-\infty)$, and the geometric mean to be $m(0)$. After all, what other particularly nice number falls between -1 and 1?

Problem 7. There is a peculiar species of worm which can either climb up 7 feet at once, or climb down 5 feet at once, but no more or less at any given step. What is the shortest pole that he can both climb up and climb down?

(A) 10 (B)$^\heartsuit$ 11 (C) 12 (D) 13 (E) more

Solution. First solution. It is possible for the worm to climb and descend an 11-foot high pole: $0 \nearrow 7 \searrow 2 \nearrow 9 \searrow 4 \nearrow 11 \searrow 6 \searrow 1 \nearrow 8 \searrow 3 \nearrow 10 \searrow 5 \searrow 0$. If the pole were any shorter, then the worm would be stuck at or before $0 \nearrow 7 \searrow 2 \nearrow 9 \searrow 4$.

Second solution. In general, assume the worm can climb up x or down y at a time. The shortest pole that the worm can climb up and down is $x+y-\gcd(x,y)$. To prove this, note that by dividing through by $\gcd(x,y)$, we need only consider the case when x and y are relatively prime; moreover, we may assume $x > y$ by symmetry.

To see that the worm cannot climb up and down a shorter pole, consider the worm's height modulo y. The worm starts at height 0 and ends at height 0 if we are to consider a single trip all the way up and then down the pole. The height modulo y either doesn't change (if he goes down y) or goes up by x. Because x and y are relatively prime, this means all residues modulo y must be hit. However, on pole shorter than $x+y-1$, the worm cannot reach height $x-1 \mod y$, and so cannot hit one of the desired residue classes.

On the other hand, he can easily climb up and down a pole of height $x+y-1$. Indeed, suppose he starts at the top of the pole. By the discussion above, he can hit every residue class modulo y (if he always takes as many down moves as possible before an up move, he can never get stuck), and so in particular can hit $0 \bmod y$; from there, a few steps down gets him down. Flipping the argument, he can hit every residue class modulo x, so in particular he can hit $0 \bmod x$; from there, a few steps up gets him up.

Problem 8. Let n be the greatest number that is the product of some positive integers (possibly not distinct), such that the sum of these integers is 2009. Find the last digit of n.

(A) 0 or 1 (B) 2 or 3 (C) 4 or 5 (D)$^\heartsuit$ 6 or 7 (E) 8 or 9

Solution. Consider the positive integers in the product. It is not optimal to have any 1s, because replacing 1 and k by $k+1$ increases the product. Similarly, numbers greater than 4 are not optimal because $k > 4$ may be replaced by 2 and $k-2$; even 4 may be replaced by two 2s without changing the product. Thus we may assume there are only 2s and 3s. However, $3^2 > 2^3$ so there should be at most two 2s. Thus, the optimal product is $n = 2 \cdot 3^{669}$. The last digit of 3^k is 3 when k is one more than a multiple of four. Thus the last digit of n is 6.

Problem 9. A group of hikers went on a 3.5-hour hike. In any consecutive one-hour period during their hike, they covered exactly two miles. What is the most distance they could have covered (in miles)?

(A) 6.5 (B) 7 (C) 7.5 (D)$^\heartsuit$ 8 (E) None of the above

Solution. They covered at most 8 miles because they covered at most 2 miles in each of the time periods $[0,1]$, $[1,2]$, $[2,3]$, and $[2.5,3.5]$, but that even double-counted the progress in $[2.5,3]$.

It is, however, possible to cover 8 miles in the following manner: the hikers covered two miles at uniform speed in each of the intervals $[0,0.5]$, $[1,1.5]$, $[2,2.5]$, $[3,3.5]$ but rested and made no progress during each of the intervals $[0.5, 1]$, $[1.5, 2]$, and $[2.5, 3]$.

2 Medium Problems

Problem 10. If a star is born in 2009, how many times will it happen that both its age and the year are perfect squares?

(A) 1 (B) 2 (C)$^\heartsuit$ 3 (D) 4 (E) more

Solution. We need to solve the equation

$$2009 + a^2 = b^2, \quad \text{i.e.} \quad 2009 = a^2 - b^2 = (a-b)(a+b)$$

For every way to factor 2009 as $n \cdot m = (a-b)(a+b)$ with $n < m$, there will be $a = (n+m)/2$ and $b = (m-n)/2$. (Note here that both n and m are odd, so $n \pm m$ will be divisible by 2.)

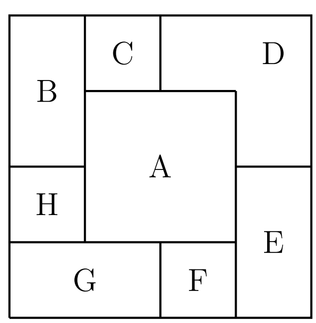
So the answer is half the number of divisors of 2009 (because we only consider divisors n with $n < m = 2009/n$.) Factor

$$2009 = 2025 - 16 = 45^2 - 4^2 = 41 \cdot 49 = 41 \cdot 7^2$$

Every divisor has the form $41^a \cdot 7^b$ with $a = 0, 1$ and $b = 0, 1, 2$. So the number of all divisors is $2 \cdot 3 = 6$, and half of that is 3.

The ages will be: 16, 1,960, and 1,008,016. So this must literally be a *star*, not a human!

Problem 11. Eight square tissues of the same size were placed on the table, one by one, to form the picture shown below. In the order of placement, which was the tissue marked B?



(A) the second (B) the third (C) the fourth (D)$^\heartsuit$ the fifth (E) the sixth

Solution. By examining local parts of the picture, we can see that A was placed last and D placed second to last. We may also conclude that C was placed after

B but before D. Similarly F was placed after E but before G. Finally H was placed after G but before B. Following this chain of conclusions, the following is the order of placement:

E F G H B C D A

So tissue B was fifth.

Problem 12. How many two-digit numbers A have the property that the square of the sum of the digits of A equals the sum of the digits of A^2?

(A) 6 (B) 7 (C) 8 (D)$^\heartsuit$ 9 (E) None of the above

Solution. First solution. Note that $A^2 < 99^2 = 9801 < 9999$. Therefore, the sum of digits of A^2 is less than $9 \cdot 4 = 36$. Since it equals the square of the digits of A, the sum of digits of A is less than $\sqrt{36} = 6$, i.e., is less than or equal to 5. There are now 15 numbers to check: 10, 11, 12, 13, 14, 20, 21, 22, 23, 30, 31, 32, 40, 41, 50, from which 9 numbers work: 10, 11, 12, 13, 20, 21, 22, 30, 31.

Second solution. Denote by $S(n)$ the sum of the digits of n. Note that when adding two numbers, either the digits add up or there is a carry of 1 on the left, and every time this happens the sum of digits drops by 9. Thus, $S(n) \le n$ since $n = 1 + \cdots + 1$, and equality holds if and only if n is a one-digit number. Also, note that $S(10n) = S(n)$.

Let the two-digit number be $A = 10a + b$. Then

$$\begin{aligned} S(A^2) &= S((10a+b)^2) = S(100a^2 + 10 \cdot 2ab + b^2) \le S(100a^2) + S(10 \cdot 2ab) + S(b^2) \\ &= S(a^2) + S(2ab) + S(b^2) \le a^2 + 2ab + b^2 = (a+b)^2 = S(A)^2 \end{aligned}$$

and equality holds if and only if a^2, $2ab$, and b^2 are 1-digit numbers. Thus, it is necessary and sufficient that $1 \le a \le 3$, $0 \le b \le 3$ and $ab \le 4$. This gives the 9 possibilities 10, 11, 12, 13, 20, 21, 22, 30, 31.

Problem 13. What is the slope of the line that bisects the angle in the first quadrant formed by the lines $y = 0$ and $y = 2x$?

(A) $\dfrac{\sqrt{5}+1}{2}$ (B) 1 (C)$^\heartsuit$ $\dfrac{\sqrt{5}-1}{2}$ (D) $\dfrac{1}{2}$ (E) $\dfrac{1}{3}$

Solution. If θ is the angle formed by the two lines, then $\tan\theta = 2$. Letting $\phi = \theta/2$, we have $\dfrac{2\tan\phi}{1-\tan^2\phi} = 2$, and so $\tan^2\phi + \tan\phi - 1 = 0$. Thus, $\tan\phi = \dfrac{-1 \pm \sqrt{5}}{2}$. We discard the negative solution, which is the slope of the perpendicular to the angle bisector.

Problem 14. How many positive integers n less than 2009 have the property that mn is **not** divisible by 2009 for **every** positive integer $m < 2009$?

(A) 1674 (B)$^\heartsuit$ 1680 (C) 1722 (D) 1960 (E) None of the above

Solution. Acceptable integers n must be relatively prime to 2009. Since $2009 = 7^2 \cdot 41$, we see that n cannot be divisible by 7 or by 41. There are $2009 \cdot \dfrac{6}{7} \cdot \dfrac{40}{41} = 7 \cdot 6 \cdot 40 = 1680$ eligible values of n.

In general, the number of positive integers less than or equal to n which are relatively prime to n is called Euler's totient function of n, written $\varphi(n)$. It is of particular interest in number theory and has many nice properties. For example, $\varphi(p) = p - 1$ for prime p and $\varphi(mn) = \varphi(m)\varphi(n)$ if m and n are relatively prime. The formula used above is a special case of the identity

$$\varphi(n) = n \cdot \prod_{\substack{p \mid n \\ p \text{ prime}}} \frac{p-1}{p}.$$

Problem 15. What is the distance between a vertex and the center of a regular tetrahedron of side one?

(A) $\frac{2\sqrt{6}}{9}$ (B)$^\heartsuit$ $\frac{\sqrt{6}}{4}$ (C) $\frac{3\sqrt{3}}{8}$ (D) $\frac{\sqrt{6}}{3}$ (E) None of the above

Solution. First solution. Let A and B be vertices, X the center, and H the center of the face opposite A. The distance from the center to a vertex, AX, is three-fourths of the height of the tetrahedron, AH. (This is analogous to the statement that for an equilateral triangle, the distance from a vertex to the center is two-thirds of the height.) To find the height, we use the Pythagorean theorem $AB^2 = AH^2 + HB^2$. Since $AB = 1$, we now need to find HB, which is the aforementioned distance from a vertex to the center of an equilateral triangle.

The height of an equilateral triangle of side 1 is $\frac{\sqrt{3}}{2}$, so $HB = \frac{2}{3}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}$, so $AH = \sqrt{1 - \frac{1}{3}} = \sqrt{\frac{2}{3}}$. Finally, three-fourths of this is $AX = \frac{3}{4} \cdot \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{4}$.

Second solution. Consider a tetrahedron of side $4\sqrt{2}$ embedded in 4-dimensional space so that its vertices are $(4,0,0,0)$, $(0,4,0,0)$, $(0,0,4,0)$, and $(0,0,0,4)$. (The tetrahedron lies in the hyperplane $x + y + z + w = 4$.) The center of this tetrahedron is clearly $(1,1,1,1)$ and so the distance to a vertex is $\sqrt{3^2 + 1^2 + 1^2 + 1^2} = 2\sqrt{3}$. We scale down our answer for a unit tetrahedron to get $\frac{2\sqrt{3}}{4\sqrt{2}} = \frac{\sqrt{6}}{4}$.

Problem 16. What is the sum of the distances from the point $(1,1)$ to the sides of the equilateral triangle with vertices at $(0,0)$, $(4,0)$, and $(2, 2\sqrt{3})$?

(A) 2 (B) $3(2\sqrt{3} - 1)$ (C)$^\heartsuit$ $2\sqrt{3}$ (D) 4 (E) $4\sqrt{3}$

Solution. The sum S doesn't depend on the point in the interior. Indeed, let h_1, h_2, and h_3 be the three distances, and split up the equilateral triangle into three triangles with base 1 and heights h_1, h_2, and h_3. Then by the $\frac{1}{2}bh$ formula for the area of a triangle, the area of the whole equilateral triangle $4\sqrt{3}$ is equal to $2h_1 + 2h_2 + 2h_3$, and so in particular $h_1 + h_2 + h_3 = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$.

Problem 17. If the side lengths of a right triangle are all integers and one is 2009, find their largest possible sum.

(A) 4410 (B) 16072 (C)$^\heartsuit$ 4038090 (D) 4040099 (E) there is none.

Solution. To get the largest sum, we want the given length to be one of the legs, not the hypotenuse. If one leg has length A (in our case an integer), then we want to maximize $x + y$, where x and y are integers satisfying $A^2 + x^2 = y^2$. Since $A^2 = y^2 - x^2 = (y - x)(x + y)$, to maximize $x + y$ we want to set $y - x = 1$ and write $A^2 = 2x + 1$, and then the sum of the side lengths will be $A + 2x + 1 = A^2 + A = A(A + 1) = (2009)(2010) = 4,038,090$.

Problem 18. 2009 kittens are sleeping in their individual crates, numbered 1 through 2009. A veterinary assistant decides to open all 2009 doors in order while the cats are asleep. He then goes back to the beginning and closes all the even-numbered doors. He then goes back to the beginning and changes every door divisible by three (i.e., if it's open, he closes it, and if it's closed, he opens it). He then continues this process for every integer $k \le 2009$, so that on the last trip, he changes precisely the 2009^{th} door. When the kittens awake in the morning, what is the largest numbered crate whose kitten will roam free?

(A) 1 (B) 1024 (C)$^\heartsuit$ 1936 (D) 2009 (E) None of the above

Solution. The position of a door changes once for each distinct factor of its assigned number. Since a door is opened with factor 1, it will end up open precisely when there are an odd number of distinct factors. The positive integers with an odd number of distinct factors are the perfect squares. The largest perfect square less than 2009 is $44^2 = 1936$.

Problem 19. Rachelle has 100 letters addressed to 100 different people and must place them in corresponding envelopes. Out of boredom, she puts one letter at random in each envelope. What is the expected number of letters that end up in correct envelopes?

(A) 0 (B) $1/e$ (C)$^\heartsuit$ 1 (D) 2 (E) 3

Solution. First solution: Let X_i be the random variable that is 1 when the i^{th} letter is put in the correct envelope and 0 otherwise. Clearly the expected value of X_i is $E(X_i) = 1/100$. Since $E(X + Y) = E(X) + E(Y)$ for any random variables X and Y, we have

$$E(X_1 + X_2 + \cdots + X_{100}) = 100 \cdot \frac{1}{100} = 1\,,$$

but $X_1 + \cdots + X_{100}$ gives precisely the number of letters in the correct envelopes.

Second solution: We give an inductive proof that the answer is always 1. Clearly, with $n = 1$ letters, the answer is 1. Suppose we know that the expected number of correct letters is 1 for all $n \le k$. With $k+1$ letters, let Y be the event that letter $k+1$ is placed in the correct envelope; $P(Y) = 1/(k+1)$. Let X be the number of letters in the correct envelopes. Then

$$E(X) = E(X \mid Y)P(Y) + E(X \mid \text{not } Y)P(\text{not } Y)\,.$$

By the induction hypothesis, $E(X \mid Y) = 1 + 1 = 2$.

On the other hand, if letter $k+1$ is put in the wrong envelope, say the ℓ^{th}, letters 1 through k must be put in envelopes 1 through k, with $k+1$ replacing ℓ. Since letter ℓ cannot possible end up in the correct envelope, $E(X \mid \text{not } Y) = 1 - 1/k$.

Thus,

$$E(X) = 2 \cdot \frac{1}{k+1} + \left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{k+1}\right) = \frac{2}{k+1} + \frac{k-1}{k} \cdot \frac{k}{k+1} = 1\,.$$

Problem 20. What is the product of the lengths of all the diagonals of a regular octagon with sidelength 1? (A diagonal is a line segment connecting two vertices that are not adjacent.)

(A) $\dfrac{(2+\sqrt{2})^{10}}{1024}$ (B) $256(2+\sqrt{2})^4$ (C)$^\heartsuit$ $\dfrac{(2+\sqrt{2})^{14}}{4}$ (D) $\dfrac{(2+\sqrt{2})^{28}}{16}$
(E) None of the above

Solution. First, consider a regular octagon inscribed in the unit circle, and consider the 5 diagonals emanating from a fixed vertex. They have length $\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, 2, $\sqrt{2+\sqrt{2}}$, and $\sqrt{2}$, respectively (using the law of cosines for the non-obvious one). Thus, the product of the lengths of the diagonals emanating from a fixed vertex is $4(2+\sqrt{2})$. Since each diagonal connects two vertices, the product of the lengths of *all* the diagonals will therefore be

$$\big(4(2+\sqrt{2})\big)^{8/2} = \big(4(2+\sqrt{2})\big)^4\,.$$

Note that the sidelength of the octagon inscribed in the unit circle is $2\sin(\pi/8)$. So to obtain an octagon with sidelength 1 we must scale everything up by a factor of $\dfrac{1}{2\sin(\pi/8)}$. Since there are 20 diagonals, the product of the lengths will therefore scale up by a factor of $\left(\dfrac{1}{2\sin(\pi/8)}\right)^{20} = \left(\dfrac{1}{4\sin^2(\pi/8)}\right)^{10}$. Recall that $4\sin^2(\pi/8) = 2\big(1-\cos(\pi/4)\big) = 2-\sqrt{2}$, so $\dfrac{1}{4\sin^2(\pi/8)} = \dfrac{2+\sqrt{2}}{2}$.

Thus, the product of the lengths of all the diagonals of the regular octagon with sidelength 1 is

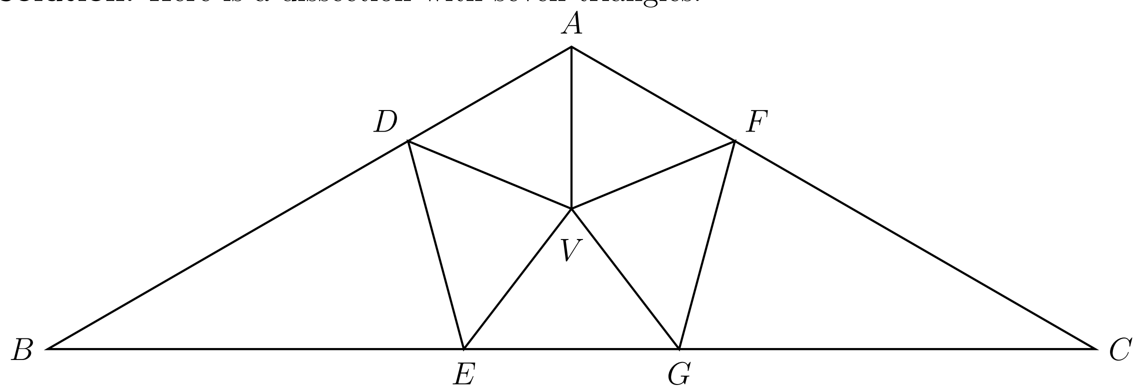
$$\big(4(2+\sqrt{2})\big)^4 \left(\frac{2+\sqrt{2}}{2}\right)^{10} = \frac{(2+\sqrt{2})^{14}}{4}\,.$$

3 Hard Problems

Problem 21. What is the least number n so that a $30°$–$30°$–$120°$ triangle can be cut into n acute triangles?

(A) 8 (B) 9 (C) 10 (D) 11 (E)$^{\heartsuit}$ None of the above

Solution. Here is a dissection with seven triangles:



It is not possible to cut a nonacute triangle into less than seven acute triangles. Indeed, consider a minimal dissection, say of a triangle ABC with nonacute angle A. One line segment must divide angle A; however, it cannot meet the other side since otherwise another nonacute triangle would be formed and our dissection would not be minimal. Therefore a vertex of the dissection lies within the triangle, say V. At least five line segments must meet at V, forming a centrally-divided pentagon. It follows there are least seven triangles because any dissection of a triangle into triangles and a single pentagon must contain at least two triangles.

Conversely, every nonacute triangle *can* be dissected into seven acute triangles. Indeed, draw the incircle and choose two tangents DE and FG to it so that BDE and CFG are acute triangles. Then the angle bisectors of the pentagon $ADEFG$ complete the dissection, because a triangle with two acute angles greater than $45°$ is acute.

Problem 22. What is the largest number of knights that may be placed on a toroidal 5×5 chessboard so that no knight attacks another? A *toroidal* chessboard is one in which the left and right edges have been identified, as well as the top and bottom edges. Thus a piece can move off the left end of the board and end up at the same height on the right end, and similarly with top and bottom.

(Recall that in chess, a *knight* can move two squares horizontally and one square vertically, or two squares vertically and one square horizontally. The knight, however, does not attack the squares along the way to its destination; thus a single knight attacks 8 squares.)

In the following picture, the knight is denoted by N and it attacks the eight numbered squares.

			8	5
6		7		
3		2		
			1	4
	N			

(A) 4 (B)$^\heartsuit$ 5 (C) 6 (D) 7 (E) None of the above

Solution. It is easy to achieve five non-attacking knights, for example by placing them all in the same row. To prove that you cannot place more, consider the following pattern on the board:

E	B	D	A	C
D	A	C	E	B
C	E	B	D	A
B	D	A	C	E
A	C	E	B	D

Each letter marks five squares, all of which attack each other. Thus you may place at most one knight within each group of identical letters.

Problem 23. What is the 57^{th} digit in the decimal expansion of $1/23$?

(A) 0 or 9 (B) 1 or 8 (C) 2 or 7 (D) 3 or 6 (E)$^\heartsuit$ 4 or 5

Solution. The decimal expansion of $1/23$ is a repeating decimal with period a divisor of 22, i.e., 2, 11, or 22. Since long division shows that the decimal starts with $0.043\ldots$, the period is not 2. If the period is 11, then the 57^{th} digit will be the 2^{nd} digit, 4. If the period is 22, the 57^{th} digit will be equal to the 13^{th} digit. By Midy's Theorem, the sum of the 2^{nd} and 13^{th} digits is 9, so the 13^{th} digit will be 5. So the 57^{th} digit is either 4 or 5. In fact, it is 5, since the period of the decimal is 22.

For reference, recall that Midy's theorem says that if a fraction a/p (p prime) has period $2n$ and so may be written as

$$\frac{a}{p} = 0.\overline{d_1 d_2 d_3 \cdots d_n d_{n+1} \cdots d_{2n}},$$

then $d_i + d_{i+n} = 9$.

Problem 24. In order to win a (tennis-like) game, one must win 3 points and also win by a margin of 2 points. (Thus, possible winning scores are 3-0, 3-1, 4-2, 5-3, etc.) If Boris wins each point with probability $p = 2/3$, what is the probability that he wins the game?

(A) 176/405 (B) 304/405 (C)$^\heartsuit$ 112/135 (D) 132/135 (E) None of the above

Solution. First solution: The probability that Boris wins by a score of 3-0 or 3-1 is $p^3 + 3p^3(1-p)$ (there are three paths to 3-1 from 0-0 that don't pass through 3-0). Now the probability that he wins after getting to 2-2 is

$$\binom{4}{2} p^2(1-p)^2 \left(p^2 + 2p(1-p)p^2 + 4p^2(1-p)^2p^2 + \dots\right) = 6p^4(1-p)^2 \frac{1}{1-2p(1-p)}.$$

Thus, the probability that Boris wins is

$$p^3 + 3p^3(1-p) + \frac{6p^4(1-p)^2}{1-2p(1-p)},$$

which equals $112/135 \approx 83\%$ when $p = 2/3$.

Second Solution: We avoid the infinite series in the previous argument by noticing that *once* the score gets to 2-2, we can focus on pairs of points played after that moment. We don't care about pairs where each players wins a point, so we're waiting for a pair when either player wins both points. The probability that Boris wins in that situation is

$$\frac{p^2}{p^2 + (1-p)^2},$$

as before.

Now we imagine a slightly different game: We play 4 points, regardless, but if the score is 2-2 we continue until someone wins as usual. We let the reader sort out why the probabilities are the same for this game. Then the probability that Boris wins will be (considering that his opponent wins 0, 1, or 2 points of the first four points played)

$$\binom{4}{0}p^4 + \binom{4}{1}p^3(1-p) + \binom{4}{2}p^2(1-p)^2 \frac{p^2}{1-2p(1-p)} = p^4 + 4p^3(1-p) + \frac{6p^4(1-p)^2}{p^2+(1-p)^2}.$$

This approach may be generalized to games of the form "first to k wins, except one must win by 2" for any k.

Problem 25. A set S of (distinct) positive integers has the property that the sum of any three of them is a prime number. What is the largest possible number of elements S can have?

(A) 3 (B)$^\heartsuit$ 4 (C) 5 (D) 6 (E) None of the above

Solution. One example of four numbers satisfying the conditions is 1, 3, 7, 9. Indeed, $1+3+7=11$, $1+3+9=13$, $1+7+9=17$ and $3+7+9=19$ are all prime numbers.

Let us show that five (or more) such numbers do not exist. Consider the remainders obtained by dividing the five numbers by 3. If there are three numbers with the same remainder then their sum is divisible by 3. If three numbers with the same remainder do not exist then there are numbers with every remainder 0, 1, and 2. Then *their* sum is divisible by 3. On the other hand, this sum is larger than 3 since the numbers are positive and distinct. So the sum is not a prime number.

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Sources. Problem 11 was borrowed from the online olympiad "Socrates" `http://www.math-on-line.com/olympiada-math/`. Problems 1, 12 and 3 are from the Moscow Mathematical Olympiad at `http://olympiads.mccme.ru/mmo/index.htm`. Problem 2 is from the Moscow Mathematical Festival at `http://olympiads.mccme.ru/matprazdnik/`. Problem 25 is from the Russian olympiad `http://olympiads.mccme.ru/mmo/okrug/okr08n.htm`. Problem 21 was proposed by Martin Gardner in his column "Mathematical Games" in *Scientific American*. Problem 8 is from the 1976 International Mathematical Olympiad.