Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes
November 8, 2008
Dedicated to the memory of Steve Sigur (Paideia School)

WITH SOLUTIONS

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

1 Easy Problems

Problem 1. What is the closest integer to

$$\sqrt{9! \cdot 9} + \sqrt{9!/9},$$

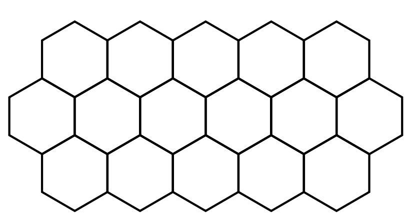
where $9! = 9 \cdot 8 \cdot 7 \cdots 1$ denotes "9 factorial"?

(A) 0 (B) π (C) 9 (D)$^\heartsuit$ 2008 (E) 10^6

Solution. This is just a fun fact for warmup (the exact value is approximately 2007.984). The intended solution was to notice that most of the answers are of the wrong order of magnitude.

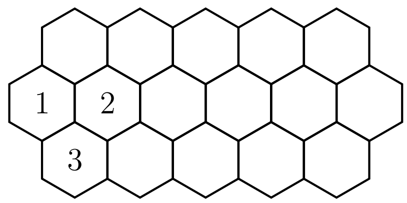
Alternate solution. Simplifying the expression gives $240\sqrt{70}$. Because $8 < \sqrt{70} < 9$, the answer is between $240 \cdot 8 = 1920$ and $240 \cdot 9 = 2160$. Note that $\sqrt{70}$ is actually very nearly $8 + \frac{11}{30}$, which yields exactly 2008 when multiplied by 240. (Equivalently, the square of the given number is $\frac{10}{9}10!$, which is near 4 million.)

Problem 2. How many ways are there to color the hexagonal regions of the diagram below with the three colors red, green, and blue so that no two adjacent regions are colored the same?



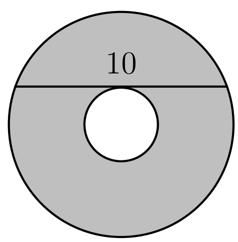
(A) 0 (B) 3 (C)$^\heartsuit$ 6 (D) 12 (E) 16

Solution. Consider the regions labeled 1, 2, and 3 in the diagram below. Note that they must all be different colors, and once their colors are chosen, the rest of the coloring is determined uniquely. Thus, there are $3! = 6$ possible colorings.



Connections. In general, determining *whether* a diagram like this is *properly* 3-*colorable* is NP-complete, while determining the *number* of colorings is $\#P$-complete. In either case, the prevailing opinion among computer scientists is that this means some large examples are intractable.

Problem 3. A chord of a circle is tangent to a smaller, concentric circle. Given that the length of the chord is 10, find the area of the donut shape ("annulus") in between the two circles.



(A) 9π (B) 16π (C)$^\heartsuit$ 25π (D) 50π (E) None of the above

Solution. Notice that there is a degree of freedom in the problem that we can exploit. Indeed, suppose the smaller "circle" actually has radius 0 and so is a point. Then the larger circle has diameter 10, and the "donut shape" is simply the area of the larger circle, which is $\pi\left(\frac{10}{2}\right)^2 = 25\pi$.

Alternate solution. Suppose the inner radius is r and the outer radius is R. Then by the Pythagorean theorem, the chord's length is equal to $2 \cdot \sqrt{R^2 - r^2}$. Thus, $R^2 - r^2 = 25$, and the area in between the circles is

$$\pi R^2 - \pi r^2 = \pi\left(R^2 - r^2\right) = 25\pi.$$

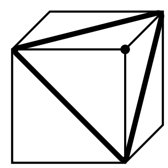
Problem 4. How many equilateral triangles can be formed from the vertices of a cube?

(A) 2 (B) 6 (C)$^\heartsuit$ 8 (D) 12 (E) 24

Solution. There are no equilateral triangles with side-lengths equal to the side of the cube or to the space ("long") diagonal. Thus, all equilateral triangles have the face diagonals as their sides.

Each of the twelve face diagonals participates in two equilateral triangles, but each triangle has three sides, so there are $12 \cdot \frac{2}{3} = 8$ triangles total. It is interesting to note that some of the vertices of a cube form a regular tetrahedron. How many are there?

Alternate solution. Each triangle can be naturally identified with a single vertex, namely the vertex that is adjacent to each of the vertices used in the triangle (see the diagram below). There are 8 vertices, and thus 8 equilateral triangles.



Problem 5. What is x if $x \geq 1$ and

$$x^{\log_2 x} = 16,$$

where $\log_2 x$ denotes the logarithm of x to the base 2?

(A) 2 (B)$^\heartsuit$ 4 (C) 6 (D) 8 (E) None of the above

Solution. This problem can easily be solved by trial and error. To find the answer directly, take the base-2 logarithm of both sides to get

$$(\log_2 x) \cdot (\log_2 x) = \log_2 16 = 4,$$

so $\log_2 x = 2$ and $x = 2^2 = 4$. (If $x < 1$, then we could also have $\log_2 x = -2$ and $x = 1/4$.)

Connections. The function $x^{\log x}$ is an example of a function that is super-polynomial (grows faster than any polynomial, like x^n) yet sub-exponential (grows slower than any exponential, like a^x).

Problem 6. Four pencils are labeled with the names of the four people on a complete UGA math tournament team (all the names are different). How many ways are there to distribute each pencil to a team member so that nobody has their matching pencil? Note that a team member may be given none, one, some, or all of the pencils.

(A) 9 (B) 16 (C) 27 (D)$^\heartsuit$ 81 (E) None of the above

Solution. Consider any single pencil. In order to satisfy the desired condition, it may be given to one of three team members (anyone but the corresponding person). We must make four independent choices (one for each pencil) among three alternatives, so there are $3^4 = 81$ total possible combinations.

Problem 7. Boris has 20 stones in a single pile, and he is trying to split them up so that each stone ends up in a pile by itself. Every time he splits a pile into two new sub-piles, one of size x and the other of size y, he gets $x \cdot y$ points added to his "score". If Boris's initial score is 0, what's the largest final score he can attain?

(A)$^\heartsuit$ 190 (B) 200 (C) 210 (D) 300 (E) 400

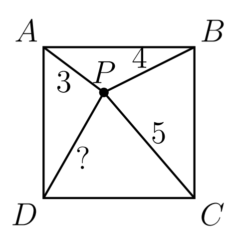
Solution. Imagine a thread connecting every pair of stones, and that whenever Boris splits a pile, the threads connecting stones in different piles are cut. Then Boris obtains exactly one point for each thread he cuts. It follows that his score is independent of the manner in which he splits piles, and is the number of threads,

$$\binom{20}{2} = \frac{20 \cdot 19}{2} = 190.$$

Alternate solution. If we examine small cases (n stones instead of 20), then we see that Boris's score seems to be independent of the splitting order and is always $\binom{n}{2}$. (Indeed, the scores for an initial pile of 1, 2, 3, and 4 stones are 0, 1, 3, and 6 respectively, which we recognize as the first few triangular numbers.) By induction, we can prove that this is always the case. Indeed, if we split a pile of n into piles of x and y, we see that the resulting score is

$$x \cdot y + \binom{x}{2} + \binom{y}{2} = \binom{x+y}{2} = \binom{n}{2}.$$

Problem 8. A point P is chosen inside a square $ABCD$ so that $AP = 3$, $BP = 4$, and $CP = 5$. Find DP.



Note: this diagram is not to scale.

(A) 2 (B) 3 (C) 4 (D) 5 (E)$^\heartsuit$ None of the above

Solution. If we conveniently "forget" that $ABCD$ is a square and instead assume it is a rectangle, there is a degree of freedom that we can exploit here. Suppose that

P actually lies on AB. Then $BC = 3$ since PBC is a 3–4–5 right triangle. Since $AD = BC = 3$, it follows that $DP = 3\sqrt{2}$ by the Pythagorean theorem. To actually show that the answer is the same for all rectangles, including the square case, we must solve the problem differently. (However, it's worth noting that deforming the picture in different ways also gives the same answer. This supports the hypothesis that it does not matter whether the rectangle is actually a square.)

Alternate solution. Suppose that when one projects P onto AB to obtain P', we have $AP' = w$ and $P'B = x$. Similarly, suppose that if we project P onto BC to obtain P'', we have $BP'' = y$ and $P''C = z$. Then we are given that

$$\begin{aligned} AP^2 &= w^2 + y^2 = 3^2, \\ BP^2 &= x^2 + y^2 = 4^2, \\ CP^2 &= x^2 + z^2 = 5^2, \end{aligned}$$

and are asked to find $DP = \sqrt{w^2 + z^2}$. But adding the first and last equations, while subtracting the second, gives $w^2 + z^2 = 3^2 + 5^2 - 4^2 = 18$, so $DP = \sqrt{18} = 3\sqrt{2}$.

Note that if you actually solve for the variables, you obtain the rather ugly values

$$\sqrt{\frac{257 + 16\sqrt{14}}{65}},\ 2\sqrt{\frac{178 + 4\sqrt{14}}{65}},\ 2\sqrt{\frac{82 - 4\sqrt{14}}{65}},\ \text{and}\ \sqrt{\frac{913 - 16\sqrt{14}}{65}}$$

for w, x, y, and z, respectively, so the approaches above are recommended.

Problem 9. How many digits are there in the first positive multiple of 6 that contains only the digits 0 and 1 (in its base 10 representation)?

(A) 3 (B)$^\heartsuit$ 4 (C) 5 (D) 6 (E) None of the above

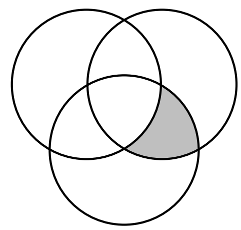
Solution. Such a number must be even, so it must end with a 0. By the standard test for divisibility by three, the number of 1s must be divisible by three. Thus the smallest number is 1110, which has four digits.

Connections. In general, for any number k, one can prove that some multiple of k consists only of the digits 0 and 1. Indeed, consider all of the numbers of the form $1, 11, 111, \ldots$. By the pigeonhole principle, some two of these leave the same remainder after division by k. (There are only k possible remainders, but infinitely many such numbers.) But then their difference $(11\cdots11) - (1\cdots1) = 11\cdots1100\cdots00$ is divisible by k.

This is connected to the decimal expansions of $1/p$ for primes p (and other numbers, too). For example, because $1/7 = 0.\overline{142857}$, it follows that $7 \mid 111111$. Indeed, $142857 \cdot 7 = 999999 = 111111 \cdot 9$.

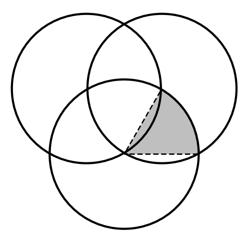
2 Medium Problems

Problem 10. In the diagram below, the three circles have unit radius and pass through one another's centers. Find the area of the shaded region.

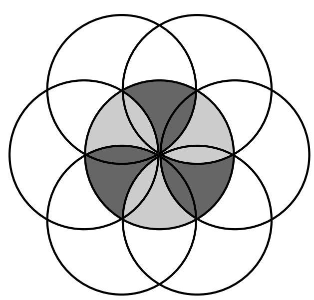


(A)$^{\heartsuit}$ $\frac{\pi}{6}$ (B) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ (C) $\frac{\pi}{3} - \sqrt{3}$ (D) $\frac{\sqrt{3}}{4}$ (E) None of the above

Solution. Consider the diagram below. It was obtained from the diagram in the problem by cutting off one piece and moving it. It follows that the desired area is simply one-sixth of the area of one of the circles, which is $\pi/6$.



Alternate solution. By making full use of symmetry, it's possible to actually see six regions like those in the problem fill up a circle:



Problem 11. How many non-negative integers less than 1000 can be expressed as

$$\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 5x \rfloor$$

for some real value of x, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x (sometimes written $[x]$ instead)?

(A) 500 (B) 545 (C) 600 (D)$^\heartsuit$ 750 (E) 1000

Solution. Suppose that x is between n and $n + \frac{1}{10}$, where n is an arbitrary integer. Then $\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 5x \rfloor = n + 2n + 5n = 8n$. Similarly, we can compute what happens in the other ten intervals of length $\frac{1}{10}$:

interval	$\lfloor x \rfloor$	$\lfloor 2x \rfloor$	$\lfloor 5x \rfloor$	sum
$n + 0.0 \leq x < n + 0.1$	n	$2n$	$5n$	$8n$
$n + 0.1 \leq x < n + 0.2$	n	$2n$	$5n$	$8n$
$n + 0.2 \leq x < n + 0.3$	n	$2n$	$5n + 1$	$8n + 1$
$n + 0.3 \leq x < n + 0.4$	n	$2n$	$5n + 1$	$8n + 1$
$n + 0.4 \leq x < n + 0.5$	n	$2n$	$5n + 2$	$8n + 2$
$n + 0.5 \leq x < n + 0.6$	n	$2n + 1$	$5n + 2$	$8n + 3$
$n + 0.6 \leq x < n + 0.7$	n	$2n + 1$	$5n + 3$	$8n + 4$
$n + 0.7 \leq x < n + 0.8$	n	$2n + 1$	$5n + 3$	$8n + 4$
$n + 0.8 \leq x < n + 0.9$	n	$2n + 1$	$5n + 4$	$8n + 5$
$n + 0.9 \leq x < n + 1.0$	n	$2n + 1$	$5n + 4$	$8n + 5$

(After this, if $n + 1.0 \leq x < n + 1.1$, the sum is $8n + 8$.) It follows that the only obtainable numbers are those that leave a remainder of 0, 1, 2, 3, 4, or 5 upon division by 8 (and conversely, all such numbers *are* obtainable). This is $\frac{6}{8} = \frac{3}{4}^{\text{th}}$s of all numbers, so our answer is $1000 \cdot \frac{3}{4} = 750$.

Problem 12. What are the last two digits of

$$2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}}?$$

Recall that x^{y^z} means $x^{(y^z)}$. Therefore, if $f(1) = 2$ and $f(n) = 2^{f(n-1)}$ for $n > 1$, the number above is $f(10)$.

(A) 16 (B)$^\heartsuit$ 36 (C) 56 (D) 76 (E) 96

Solution. Writing down the last two digits of the powers 2^n, one discovers that they are periodic with period 20, ignoring the first two powers:

$$01, 02, 04, 08, 16, 32, 64, 28, 56, 12, 24, 48, 96, 92, 84, 68, 36, 72, 44, 88,$$
$$76, 52, 04, \ldots$$

So, to find the answer, we only need to know the remainder of $2^{2^{\cdot^{\cdot^{\cdot}}}}$ modulo 20. Using the above table, one discovers that the powers 2^n mod 20 are periodic with period 4, again ignoring the first two powers:

$$01, 02, 04, 08,$$
$$16, 12, 04, \ldots$$

But clearly a "power-tower" of at least two 2s is divisible by 4, so a tower that is one taller is congruent to $2^4 \equiv 16 \pmod{20}$. Finally, a tower one taller than that is congruent to $2^{16} \equiv 36 \pmod{100}$. Thus the final two digits of any tower of at least four 2s are 36.

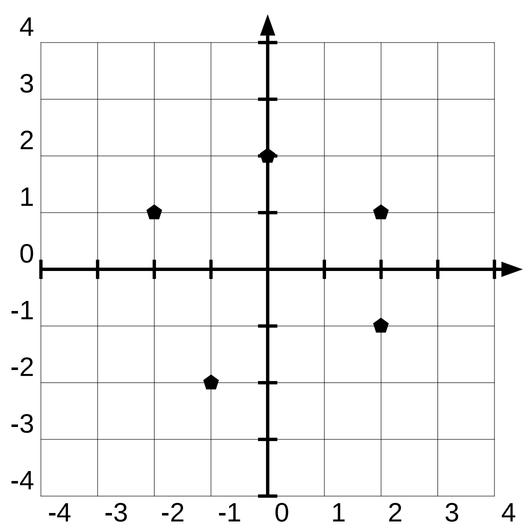
Connections. By modifying this proof slightly (requiring the use of the generalization of Fermat's Little Theorem involving Euler's totient function), one can show that any sufficiently large tower

$$k^{k^{\cdot^{\cdot^{\cdot}}}}$$

is eventually constant modulo m for any fixed k and m. Moreover, it becomes constant rather quickly. (In the problem above, one might be able to compute exact values of small towers of 2s; they are 2, 4, 16, and then 65536, which some people have memorized for its usefulness in computer programming or because of the adjacent Fermat prime 65537. From that point on, the final two digits are constant, namely 36.)

Problem 13. The midpoints of the sides of a (not necessarily convex) pentagon are, in order,

$$(2,1) \quad (2,-1) \quad (-1,-2) \quad (-2,1) \quad (0,2)$$



Which of the following was a vertex of the pentagon? (The midpoints are marked in the grid above.)

(A) (0,0) (B) (0,1) (C) (1,0) (D)$^\heartsuit$ (1,1) (E) (1,2)

Solution. Consider the vertices a, b, c, d, e of the pentagon as vectors. Then for the side connecting a and b, we are given the midpoint $m_{ab} = \frac{a+b}{2}$. From this and the four other analogous expressions, we can recover

$$a = \frac{a+b}{2} - \frac{b+c}{2} + \frac{c+d}{2} - \frac{d+e}{2} + \frac{e+a}{2} = m_{ab} - m_{bc} + m_{cd} - m_{de} + m_{ea}.$$

(If this expression seems random, note that the sum of all of the midpoint values gives $a+b+c+d+e$, so subtracting off $(b+c)+(d+e)=2m_{bc}+2m_{de}$ will leave a. This is precisely the sum of m_{ab}, m_{cd}, and m_{ea} minus the sum of m_{bc} and m_{de}.)

Thus, we can recover the five original vertices as $(1,1)$, $(3,1)$, $(1,-3)$, $(-1,-1)$, and $(-1,3)$.

Alternate solution. A perhaps easier solution is to simply check the possible answers by successively reflecting them over the provided midpoints and seeing whether or not after five reflections, the pentagon closes up on itself.

Problem 14. Evaluate

$$\sqrt{33 \cdot 34 \cdot 35 \cdot 36 + 1}.$$

(A) 1089 (B) 1091 (C) 1190 (D) 1191 (E)$^\heartsuit$ None of the above

Solution. Suppose the problem asked to evaluate

$$\sqrt{(x-1) \cdot x \cdot (x+1) \cdot (x+2) + 1}$$

instead. Then we can expand the interior of the square root as

$$(x-1) \cdot x \cdot (x+1) \cdot (x+2) + 1 = x^4 + 2x^3 - x^2 - 2x + 1 = (x^2 + x - 1)^2.$$

In our case, $x = 34$ and so we obtain $34^2 + 34 - 1 = 1189$.

Alternate solution. Let $I = 33 \cdot 34 \cdot 35 \cdot 36 + 1$. Working mod 9, $I \equiv 1$, while $A^2 \equiv D^2 \equiv 0$ and $B^2 \equiv C^2 \equiv 1$. This eliminates answers A and D. Next, working mod 100, $I \equiv 33 \cdot 34 \cdot 7 \cdot 18 \cdot 10 + 1$ and $33 \cdot 34 \cdot 7 \cdot 18 \equiv 2 \pmod{10}$, so $I \equiv 21 \pmod{100}$. But $B^2 \equiv 91^2 \equiv 9^2 \equiv 81 \pmod{100}$ and $C^2 \equiv 0 \pmod{100}$, so none of A, B, C, D equals $\sqrt{I}$.

Problem 15. Suppose

$$\begin{aligned} a + 4b + 9c + 16d + 25e &= -13, \\ 4a + 9b + 16c + 25d + 36e &= -8, \\ 9a + 16b + 25c + 36d + 49e &= 3. \end{aligned}$$

Find

$$16a + 25b + 36c + 49d + 64e.$$

(A) 2 (B) 8 (C) 16 (D) 18 (E)$^\heartsuit$ 20

Solution. By taking the difference between consecutive rows, we find that

$$\begin{aligned} 3a + 5b + 7c + 9d + 11e &= 13 - 8 = 5, \\ 5a + 7b + 9c + 11d + 13e &= 8 - (-3) = 11. \end{aligned}$$

By taking the differences between consecutive rows again, we find that

$$2a + 2b + 2c + 2d + 2e = 11 - 5 = 6.$$

Adding this back to the last row of the previous table, we find that

$$7a + 9b + 11c + 13d + 15e = 11 + 6 = 17.$$

Finally, we add this back again to the last row given in the problem statement to find that

$$16a + 25b + 36c + 49d + 64e = 3 + 17 = 20.$$

Alternate solution. We can find specific values of a, b, c, d, e that work by supposing $d = e = 0$ and solving the resulting system of equations to get $a = 0$ as well, while $b = 8$ and $c = -5$. It follows that

$$25b + 36c = 25 \cdot 8 - 36 \cdot 5 = 200 - 180 = 20.$$

Third solution. Let $f(x) = a(x+1)^2 + b(x+2)^2 + c(x+3)^2 + d(x+4)^2 + e(x+5)^2$. Then f is a quadratic polynomial, say $f(x) = px^2 + qx + r$. Since $f(0) = 13$, $r = 13$. Since $f(1) = 8$ and $f(2) = 3$, we have $p + q = 5$ and $4p + 2q = 16$, so $p = 3$ and $q = 2$. Therefore, $f(x) = 3x^2 + 2x - 13$, and $f(3) = 20$.

Problem 16. How many of the four 9-digit numbers

111,222,333
111,333,222
222,333,111
333,222,111

are divisible by 13?

(A) 0 (B) 1 (C)$^\heartsuit$ 2 (D) 3 (E) 4

Solution. There is a little-known test for divisibility for 7, 11, and 13 simultaneously. Divide the candidate number into groups of three, like the standard grouping by commas. Then alternately subtract and add these groups. If the result is divisible by 13 (or 7 or 11), then so was the original number. The reason this test works is precisely analogous to why the usual test for divisibility by 11 works; that is, we have $7 \cdot 11 \cdot 13 = 1001 = 10^3 + 1$.

When we apply this to this problem at hand, we see immediately that because $111 + 222 = 333$, the second and third numbers are divisible by 13. The other two are divisible by 13 if and only if 222 is as well, but $222 = 2 \cdot 3 \cdot 37$, so they are not.

Of course, this problem may be solved by trial division as well. It is generally useful to be able to quickly compute a large number n modulo a small number k quickly using a digit-by-digit approach, and without actually computing the quotient.

Alternate solution. You can apply another test for divisibility for 13. Take four times the last digit of a number and add it to the remaining leading truncated number. The original number was divisible by 13 if and only if the new number is. For example, $507 \mapsto 50 + 4 \cdot 7 = 78 = 6 \cdot 13$, so 507 is divisible by 13. Can you figure out why this works?

Problem 17. Solve for the integer n:

$$3^5 + 54^5 + 62^5 = 24^5 + 28^5 + n^5.$$

(A) 64 (B) 66 (C)$^\heartsuit$ 67 (D) 70 (E) 77

Solution. The final digit of n^5 is the same as the final digit of n for all integral values of n. Thus, the final digit of n must be 7. We can perform an order of magnitude calculation to see that $n = 67$.

In order to be more sure, we can analyze the equation modulo 3 as well. Here again, upon division of 3, the remainders of n^5 and n are the the same. It follows that the remainder of n upon division by 3 is 1. Thus n could only be 37, 67 or 97, but 37 is clearly too small and 97 is clearly too large.

Connections. Euler's sum of powers conjecture said that if the sum of n k^{th} powers of positive integers is itself a k^{th} power, then n is at least k. This was disproven by a computer search by L. J. Lander and T. R. Parkin in 1966 with the example

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

Later, Lander, Parkin, and John Selfridge would conjecture that if $\sum_{i=1}^{n} a_i^k = \sum_{j=1}^{m} b_j^k$ where $k > 3$ and $a_i \neq b_j$ are positive integers, then $m + n \geq k$. This problem contradicts neither of these conjectures, but is the smallest example of its particular kind, discovered by the same team of Lander, Parkin, and Selfridge in 1967.

Problem 18. If $\cos\theta = \frac{1}{3}$, find $\cos 5\theta$.

(A) $\frac{1}{243}$ (B) $\frac{41}{243}$ (C) $\frac{100}{243}$ (D) $\frac{231}{243}$ (E)$^\heartsuit$ $\frac{241}{243}$

Solution. There is a trigonometric identity

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

Since in this case, $\sin^2\theta = 1 - \cos^2\theta = \frac{8}{3^2}$, we have

$$\cos 5\theta = \frac{1 - 10 \cdot 8 + 5 \cdot 8^2}{3^5} = \frac{241}{243}.$$

Of course, no one is expected to remember this identity. It can be derived by repeated applications of the sum-of-angles identities for cosine and sine, but the easiest way to derive it is to raise Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

to the fifth power, using the Binomial Theorem to obtain

$$\begin{aligned} e^{i\cdot 5\theta} = \left(e^{i\theta}\right)^5 &= \left(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\right) \\ &\quad + i\left(5\cos^4\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right). \end{aligned}$$

By Euler's identity applied to the left-hand side, it follows that the real part of the right-hand side is $\cos 5\theta$, as desired.

Alternate solution. It is possible to avoid complex numbers by using rotation matrices. (This is equivalent, as the complex number $a + bi$ can be identified with the 2×2 real matrix $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$.) Let A be the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1/3 & -\sqrt{8}/3 \\ \sqrt{8}/3 & 1/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & -\sqrt{8} \\ \sqrt{8} & 1 \end{pmatrix}.$$

Then

$$A^2 = \frac{1}{9}\begin{pmatrix} -7 & -2\sqrt{8} \\ 2\sqrt{8} & -7 \end{pmatrix} \qquad \text{and so} \qquad A^4 = \frac{1}{81}\begin{pmatrix} 17 & 28\sqrt{8} \\ * & * \end{pmatrix},$$

so

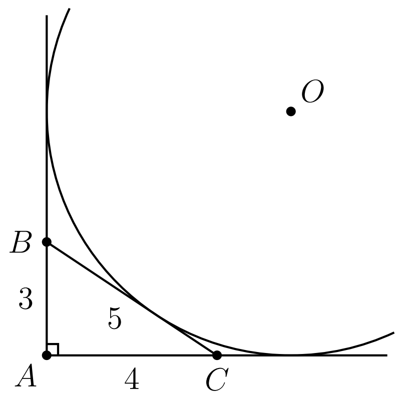
$$\begin{pmatrix} \cos 5\theta & -\sin 5\theta \\ \sin 5\theta & \cos 5\theta \end{pmatrix} = A^5 = A^4 \cdot A = \frac{1}{243}\begin{pmatrix} 17 + 28\cdot 8 & * \\ * & * \end{pmatrix},$$

and $\cos 5\theta = \dfrac{241}{243}$.

Connections. It's interesting to note that the answer, approximately equal to 0.992, is very close to 1! This is not entirely accidental. As a warmup, consider three pentagons lying in the same plane and sharing a single vertex. Their angles, each equal to 108°, together make up 324°, short of a full circle by 36°. If we want them to meet, then we have to fold up into the third dimension. By doing this and continuing the resulting pattern, we obtain a dodecahedron, a regular polyhedron made up of twelve pentagons. Because 36° is the smallest remaining angle one can obtain with triangles, squares, or pentagons, the resulting polyhedron is the largest regular polyhedron by volume if the edge length is constant.

It so happens that the dihedral angle of a regular tetrahedron is $\arccos\frac{1}{3}$. (The dihedral angle is formed by the two planes of adjacent sides.) The solution to this problem suggests that five tetrahedra that share a common edge come together to almost complete a full 360°. In order to make them meet, we must "fold up into the fourth dimension". If we do this, then we get a four-dimensional regular *polychoron* (sometimes people just say *polytope* beginning with dimension 4). As with the dodecahedron, because the leftover angle is so small, the resulting polychoron is very large; indeed, it is the largest, called the 600-*cell* or *hexacosichoron* because it is composed of 600 regular tetrahedra.

Problem 19. In the figure below, the circle is tangent to the hypotenuse and the extensions of the two legs of a 3–4–5 right triangle. Find the radius of the circle.



Note: this diagram is not to scale.

(A) $2+2\sqrt{2}$ (B)$^\heartsuit$ 6 (C) 7 (D) 8 (E) None of the above

Solution. Recall the usual formula for the radius of the incircle of a triangle ABC (the incircle is tangent to the sides, but inside the triangle) with sides a, b, and c. We have $A = rs$, where A is the area, r is the inradius, and $s = \frac{a+b+c}{2}$ is the semi-perimeter. (This can be seen easily by considering the areas of triangles ABI, BCI, and CAI where I is the incenter.)

There is a general concept in triangle geometry of what happens when one flips the sign of a side-length, say mapping $a \mapsto -a$. In this case, the incircle is sent to the excircle on the side a, while the semi-perimeter is sent to $s \mapsto s_a = \frac{-a+b+c}{2}$. (Area is preserved, which one can see by considering Heron's formula for the area of triangle.) Thus, we may recover the formula for the radius of the a excircle as $A = r_a s_a$. In our case, $A = 6$ while $s_a = \frac{3+4-5}{2} = 1$, so $r_a = 6$.

Other facts from this general theory is that symmetric quantities (such as the radius of the circumcircle, or the "location" of the median) can only depend on symmetric functions of the side lengths, such as $(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$ (*c.f.* Heron's formula) or abc (*c.f.* the formula for the circumradius $2R = abc/A$). Thus we can relate the worlds of formulas (algebra) and shapes (geometry). This approach can be very fruitful, such as here where it gave us four formulas for the price of one!

Alternate solution. By modifying the standard proof that $A = rs$, we can obtain a direct proof of the above result that $A = r_a s_a$. Indeed, the area ABC is the sum of the areas ABO and ACO minus the area BCO. But by the base-height triangle formula, the area $ABO = \frac{r \cdot c}{2}$, and similarly for the other triangles. It follows that

$$ABC = ABO + ACO - BCO = \frac{r_c c + r_a b - r_a a}{2} = r_a \frac{c+b-a}{2} = r_a s_a,$$

as desired.

Third solution. It is also possible to solve this problem in coordinates. Recall that if $ax + by + c = 0$ is the *normalized* form of a line (that is, $a^2 + b^2 = 1$), then $ax_0 + by_0 + c$ is the *signed* distance from the point (x_0, y_0) to the line.

Set up a coordinate system by letting $A = (0,0)$, $B = (0,3)$, and $C = (4,0)$. Then the center of the excircle $O = (r,r)$ for some value of r, because it is then at equal distance r from the extensions of lines AB and AC. The equation of BC is $\frac{x}{4}+\frac{y}{3}-1=0$, or in normalized form $\frac{3}{5}x+\frac{4}{5}y-\frac{12}{5}=0$. (As a check, we can confirm that $A=(0,0)$ is $\frac{12}{5}$ away from line BC, which we can compute using the base-height formula for the area of ABC applied to two different sides.) Thus, the condition that O has distance r from BC is

$$r=\frac{3}{5}r+\frac{4}{5}r-\frac{12}{5},$$

which is satisfied when $r=6$.

3 Hard Problems

Problem 20. If three positive integers $a<b<c$ satisfy $a^2+b^2=c^2$ (and thus correspond to the lengths of the sides of a right triangle) and a, b, and c share no common factor other than 1, we say that a, b, c forms a *primitive* Pythagorean triple.

For example, 16, 63, 65 is a primitive triple, while 39, 52, 65 and 25, 60, 65 are *not* because they are multiples of the 3, 4, 5 and 5, 12, 13 triples, respectively. What is $b-a$ in the other primitive Pythagorean triple where the hypotenuse c is 65?

(A) 11 (B) 25 (C) 41 (D) 59 (E)$^\heartsuit$ None of the above

Solution. Each Pythagorean triple corresponds to a rational complex number of unit modulus (absolute value). For example, the triple 3, 4, 5 corresponds to the number $\frac{3+4i}{5}$ because we can compute that

$$\left|\frac{3+4i}{5}\right|^2=\frac{3^2+4^2}{5^2}=1.$$

In this notation, non-primitive triples such as 39, 52, 65 correspond to the same value as the underlying primitive triple, since $\frac{39+52i}{65}=\frac{3+4i}{5}$.

Because the product of two rational complex numbers of units modulus gives another (recall that $|zw|=|z|\,|w|$), this procedure also gives us an easy way to generate Pythagorean triples! For example,

$$\left(\frac{3+4i}{5}\right)^2=\frac{-7+24i}{25},$$

so 7, 24, 25 is a primitive Pythagorean triple. Note also that this raises the issue that each triple actually corresponds to eight different complex numbers, such as in this case $\frac{\pm 7\pm 24i}{25}$ and $\frac{\pm 24\pm 7i}{25}$.

In any case, by multiplying the complex numbers corresponding to Pythagorean triples with hypotenuses 5 and 13, we will obtain one with hypotenuse 65. Thus we obtain

$$\frac{3+4i}{5}\cdot\frac{5+12i}{13}=\frac{-33+56i}{65}$$
$$\frac{3-4i}{5}\cdot\frac{5+12i}{13}=\frac{63-16i}{65}$$

(The other possible choices in $\frac{\pm3\pm4i}{5}$ and $\frac{\pm5\pm12}{13}$ as well as ordering all produce these same two triples.) The second triple is the one given in the problem, so the first one is our desired triple. Our final answer is $56-33=23$.

Note that using this idea, we can give Pythagorean triples the structure of a *group*, which ends up being isomorphic to an infinite direct sum of infinite cyclic groups (modulo the issue of distinguishing the eight different numbers, which can be handled adequately).

Connections. Furthermore, this problem is related to primes in the Gaussian integers (complex numbers with integral real and imaginary parts). Indeed, any number that factors in the Gaussian numbers but not the usual integers, such as $5=(2+i)(2-i)$, gives rise to a Pythagorean triple by squaring one of its factors, as in $(2+i)^2=3+4i$.

Alternate solution. The primitive Pythagorean triples are parameterized by

$$(p^2-q^2, 2pq, p^2+q^2)$$

for relatively prime integers p and q exactly one of which is odd. The given Pythagorean triple corresponds to $p=8$, $q=1$. A quick check shows that $p^2+q^2=65$ only when $(p,q)=(8,1)$ or $(p,q)=(7,4)$. So the other Pythagorean triple is $(33,56,65)$ where $b-a=23$.

Problem 21. Compute

$$\frac{(10^4+2^6)\,(18^4+2^6)\,(26^4+2^6)\,(34^4+2^6)\,(42^4+2^6)}{(6^4+2^6)\,(14^4+2^6)\,(22^4+2^6)\,(30^4+2^6)\,(38^4+2^6)}.$$

(A) 53 (B)$^{\heartsuit}$ 97 (C) 181 (D) 221 (E) None of the above

Solution. The structure of the numbers here suggests an algebraic approach. Indeed, recall Sophie Germain's identity

$$x^4+4y^4=(x^2+2xy+2y^2)(x^2-2xy+2y^2),$$

or the special case useful in our problem since $4\cdot2^4=2^6=64$,

$$x^4+64=(x^2+4x+8)(x^2-4x+8)=(x(x+4)+8)\cdot(x(x-4)+8).$$

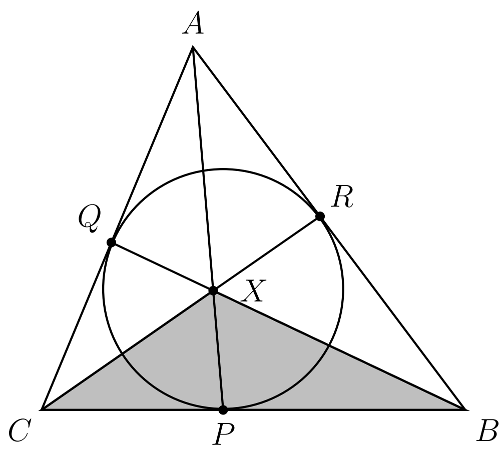
(Note that the reason x^4+4y^4 factors without complex numbers while the arguably similar x^2+4y^2 does not is related to the fact that there is "more room" or "more symmetries" in four dimensions.)

If we apply this to our problem, we see that the factor $(10^4 + 64)$ splits into $(10^2 - 4 \cdot 10 + 8) \cdot (10^2 + 4 \cdot 10 + 8)$ or equivalently $(10 \cdot 6 + 8) \cdot (10 \cdot 14 + 8)$. The factor beneath it, $(6^4 + 64)$ similarly splits into $(6 \cdot 2 + 8) \cdot (6 \cdot 10 + 8)$. Thus we see the potential for cancellation! Indeed, all but two terms cancel in a telescoping fashion until only

$$\frac{42 \cdot 46 + 8}{6 \cdot 2 + 8}$$

is left. This we can compute is $1940/20 = 97$.

Problem 22. In triangle ABC below, $CA = 5$, $AB = 6$, and $BC = 7$. The inscribed circle is tangent to the sides BC, AC, and AB at points P, Q, and R, respectively. The lines AP, BQ, and CR concur (all intersect) at the point X. Find the ratio of the area of XBC to the area of ABC.

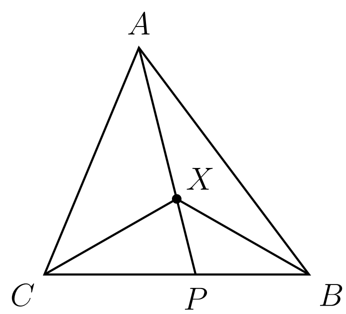


Note: this diagram is not to scale.

(A) 2/9 (B) 1/4 (C) 7/18 (D)$^\heartsuit$ 6/13 (E) None of the above

Solution. The point X is called the *Gergonne point* and has barycentric coordinates $(1/s_a, 1/s_b, 1/s_c)$, so the area of XBC is $\frac{1}{s_a}/(\frac{1}{s_a} + \frac{1}{s_b} + \frac{1}{s_c}) = \frac{1}{2}/\frac{13}{12} = \frac{6}{13}$ of the area of ABC. (Recall that $s_a = \frac{b+c-a}{2}$ and similarly $s_b = \frac{a+c-b}{2}$ and $s_c = \frac{a+b-c}{2}$.)

Let's recall the meaning of barycentric coordinates, and then derive the Gergonne point's coordinates. If X is a point in ABC, then we say it has barycentric coordinates $(x : y : z)$ if the ratio of the areas $XBC : AXC : ABX$ is $x : y : z$. As a note, we usually work unnormalized, so $(x : y : z)$ is the same point as $(kx : ky : kz)$ for any nonzero k. Equivalently, suppose that we extend AX so that it meets side BC at a point P; then the barycentric coordinates of X are $(\_ : y : z)$ if the point P divides BC in a ratio of $CP : PB = y : z$. This is equivalent because then the ratios $APC : APB = XPC : XPB = y : z$ by the base-height area formula; it follows that $AXC : AXB = (APC - XPC) : (APB - XPB) = y : z$ as well. Note also that these coordinates somewhat implicitly prove (or assume) Ceva's theorem.



This formulation gives the best system of coordinates in an arbitrary triangle. For example, the vertex A has coordinates $(1 : 0 : 0)$ while the midpoint of BC has coordinates $(0 : 1 : 1)$. The median from A thus has equation $y = z$, and so it follows that the centroid, the intersection of the medians, has coordinates $(1 : 1 : 1)$. Similarly, the usual Cevian relation for angle bisectors means that the incenter has coordinates $(a : b : c)$.

In our case, because tangents from the same point to the same circle have equal length, we know $AQ = AR$, $BP = BR$, and $CP = CQ$. By solving the simple system of linear equations $BP + PC = BC = a$, $CQ + QA = CA = b$, and $AR + RB = AB = c$, we discover that $AQ = AR = s_a$, $BP = BR = s_b$, and $CP = CQ = s_c$. This means that the ratios of the areas are like $(s_b s_c : s_a s_c : s_a s_b) = (1/s_a : 1/s_b : 1/s_c)$. (Note the slightly-surprising inverses here; without them, we would have the *Nagel point*.)

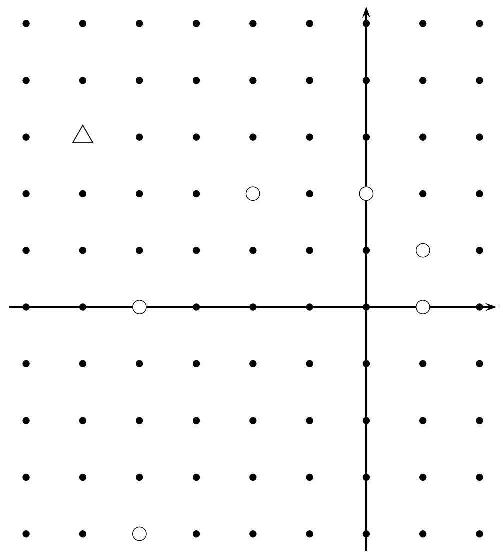
Disclaimer. In the above discussions, we have disregarded signedness of areas; all areas were assumed positive. To extend barycentric coordinates outside of the triangle, one must be more careful.

Problem 23. The complex number $-5+3i$ has a unique representation in base $1+i$, that is, as a sum of powers of $1+i$, some with coefficient 0 and some with coefficient 1. Alternatively, there exists a unique finite set P of non-negative integers so that

$$-5 + 3i = \sum_{p \in P} (1 + i)^p.$$

How many 1-coefficients are in the representation, or alternatively, how many elements are there in P?

For convenience, we have included a scratch complex plane, with the powers of $1 + i$ (which are 1, $1 + i$, $2i$, $-2 + 2i$, -4, $-4 - 4i$, $\ldots$) marked with circles. The target $-5 + 3i$ is also noted, with a triangle.



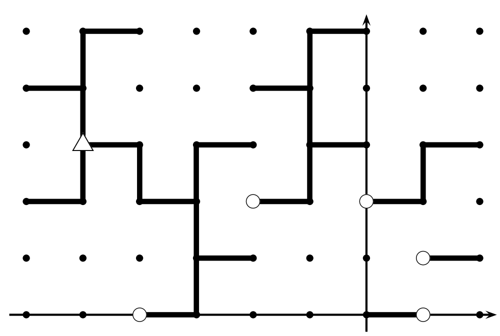
(A) 2 (B)$^\heartsuit$ 3 (C) 4 (D) 5 (E) 6

Solution. The correct representation is 11010_{1+i}, that is,

$$-5+3i = (1+i)^4 + (1+i)^3 + (1+i)^1.$$

Thus the set $P = \{1,3,4\}$ and the answer is 3.

In order to discover this, we can simply *draw* the representations of Gaussian integers (complex numbers with integral real and complex parts) in base $1+i$. If we do this, then we obtain the fractal pattern below!

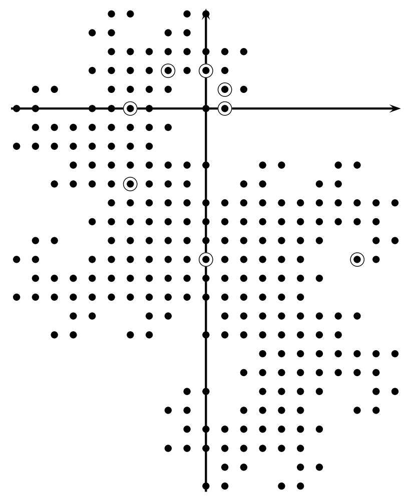


Here's what's going on: with a single digit, you can write 0 or 1. With two digits, you can write $1+i$ plus 0 or 1, which is $1+i$ or $2+i$. With three digits, you can write $2i$ plus 0, 1, $1+i$, or $2+i$. Continuing this, you get the spiraling fractal pattern illustrated. From it, we see that $-5+3i$ requires 5 digits, and with some thought we can read off the individual digits as 11010_{1+i}.

Note that "half" of the Gaussian integers can be written uniquely as a sum of powers of $1+i$. (The other half can be written as i minus a sum of powers of $1+i$.) Contrast this with other bases, such as $i-1$, where *every* Gaussian integer has a unique representation.

Connections. Donald Knuth has considered complex bases, although he focused on "quater-imaginary base $2i$" with four coefficients 0, 1, 2, and 3.

Also, the fractal under consideration is called a *dragon curve*, and is illustrated in more detail below. See `http://en.wikipedia.org/wiki/Dragon_curve` for different kinds of dragon curves.



Problem 24. Suppose that A and B are 3×3 matrices with integral entries and that $AB - BA \equiv I \pmod 3$, where I is the 3×3 identity matrix. Find $AB^3 - B^3 A$ mod 3.

(A)$^\heartsuit$ 0 (B) I (C) B (D) B^2 (E) None of the above

Solution. Because $AB = BA + I$, we can expand

$$\begin{aligned} AB^3 &= AB \cdot B^2 = BAB^2 + B^2 \\ &= B(AB \cdot B) + B^2 = B(BAB + B) + B^2 \\ &= B(B(AB) + B) + B = B(B(BA + I) + B) + B^2 \\ &= B^3 A + 3B^2, \end{aligned}$$

so $AB^3 - B^3 A = 3B^2$. But we are working modulo 3, so this is zero.

Note that it cannot be the case that $B^2 = 0$, because if so, B has rank at most 1, so AB and BA have rank at most 1. But then $AB - BA$ has rank at most 2 and cannot equal the identity, which has full rank of 3.

Alternate solution. There are many matrices A and B that work here, but perhaps the simplest pair is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In this case $B^3 = 0$, and so $AB^3 - B^3 A = A \cdot 0 - 0 \cdot A = 0$ as well.

Connections. The operation $[A, B] = AB - BA$ is called the *commutator* of A and B, and essentially it gives a measure of the extent to which commutativity fails.

Suppose we want to build the simplest possible (associative) *algebra* involving two non-commutative elements a and b. In order for things to behave normally, we require that addition be commutative and associative as usual. Multiplication will be associative as well, but obviously not commutative since we wish to have non-commutativity. Now, what is the simplest way in which commutativity can fail? We should let the commutator $[a, b]$ be the simplest possible non-zero thing, which is 1!

This equation has many connections to math and physics. For example, it is the canonical commutation relation of quantum mechanics, which says that $[x, p] = i\hbar$ where x and p are operators measuring position and momentum, respectively; this relation implies the Heisenberg uncertainty principle!

We can also interpret the equation in the context of differential operators. Indeed, suppose that D is the operation "take the derivative with respect to x" while x is the operation "multiply by x". Then for any function f,

$$([D, x])(f) = (Dx - xD)(f) = D(xf) - x(Df) = f + xf' - xf' = f,$$

so indeed $[D, x] = 1$, the identity operator. In this context, we can show that in general $[D, x^n] = nx^{n-1}$, so if p is a polynomial in x, $[D, p]$ functions as a formal derivative. (Note that the two example matrices given above correspond to these operations, except modulo 3 and modulo x^3. For example, the -1 in A above means that $D(x^2) = 2x \equiv -x \pmod 3$.)

Problem 25. When one expands $(x + y)^{2008}$ as

$$1 \cdot x^{2008} + 2008 \cdot x^{2007}y + 2015028 \cdot x^{2006}y^2 + \cdots + 1y^{2008},$$

how many of the coefficients are odd?

(A) 6 (B) 96 $(\text{C})^{\heartsuit}$ 128 (D) 250 (E) 502

Solution. Consider the first few rows of Pascal's triangle, which corresponds to the binomial coefficients that appear in the expansion of $(x + y)^n$:

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}$$

Notice that the number of odd numbers in each row is a power of 2. One can guess that this is true for every row of the triangle, and thus the answer must be 128 by elimination. We include three proofs of this fact below.

(In fact, we can say more. If there are h 1s in the binomial representation of n, then there will be 2^h odd coefficients in the expansion of $(x+y)^n$. Since $2008 = 11111011000_2$, there will be $2^7 = 128$ odd coefficients.)

First proof. Let $n = n_k 2^k + n_{k-1}2^{k-1} + \cdots + n_0$ be the binary expansion of n. We wish to find the number of coefficients in $(x+y)^n$ which are not equal to $0 \pmod 2$. Notice that $(x+y)^2 \equiv x^2 + y^2 \pmod 2$. (This is sometimes called "Freshman's Dream" or the high school student's binomial theorem. More commonly, it is known as the *characteristic* 2 *binomial theorem.*) By induction, it also follows that

$$(x+y)^4 = ((x+y)^2)^2 \equiv (x^2+y^2)^2 \equiv x^4 + y^4 \pmod 2,$$

etc. for any power of 2. Therefore, modulo 2, we have that

$$(x+y)^n = (x+y)^{n_k 2^k} \cdots (x+y)^{n_0 2^0} \equiv \left(x^{2^k} + y^{2^k}\right)^{n_k} \cdots (x+y)^{n_0},$$

where we treat a parenthesized expression raised to the 0^{th} power as simply omitted. For example,

$$\begin{aligned}(x+y)^{2008} \equiv\ & (x^{1024}+y^{1024})(x^{512}+y^{512})(x^{256}+y^{256}) \\ & (x^{128}+y^{128})(x^{64}+y^{64})(x^{16}+y^{16})(x^8+y^8).\end{aligned}$$

But consider all of the terms in the expansion of the right-hand side. They are all distinct, because binary expansions are unique (and thus so are the exponents of x). There are clearly 2^7 terms in this specific case, and 2^h terms in general.

Second proof. By Lucas's theorem, if $n = n_k 2^k + n_{k-1}2^{k-1} + \cdots + n_0$ and $i = i_k 2^k + i_{k-1}2^{k-1} + \cdots + i_0$ are the binary expansions of n and i respectively,

$$\binom{n}{i} \equiv \binom{n_k}{i_k} \cdots \binom{n_0}{i_0} \pmod 2.$$

(Note that Lucas's theorem may itself be proven using the characteristic 2 method of the previous proof. We include this solution because some people are familiar with Lucas's theorem directly, particularly for Olympiad-level number theory.) Thus, if we want the right-hand side to be $1 \mod 2$, none of the binomial coefficients $\binom{n_j}{i_j}$ can be 0. This means that if $n_j = 0$, $i_j = 0$ as well, while if $n_j = 1$, i_j can be either 0 or 1, and similarly for the other bits (binary digits). This means that if there are h 1s in the binary representation of n, there will be 2^h odd binomial coefficients.

Third proof. By writing out Pascal's triangle modulo 2 (that is, marking all of the odd coefficients with 1s), we obtain

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&0&&1&&&&& \\
&&&&1&&1&&1&&1&&&& \\
&&&1&&0&&0&&0&&1&&& \\
&&1&&1&&0&&0&&1&&1&& \\
&1&&0&&1&&0&&1&&0&&1& \\
1&&1&&1&&1&&1&&1&&1&&1
\end{array}$$

If we look carefully, we see that this is Sierpinski's triangle! Thus, we can construct the triangle by beginning with a single 1; then, at each step, we make two copies of what we have so far and place them below our current triangle, filling in the rest with zeroes. For example, the first few steps are

$$\begin{array}{c} 1 \end{array} \mapsto \begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array} \mapsto \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 0 & & 1 & \\ 1 & & 1 & & 1 & & 1 \end{array}$$

followed by the triangle above. It's clear from this construction that the number of 1s in each row is a power of 2. We begin with a single row with one 1, and every step of the process creates new rows that either have the same number of 1s as an existing row or exactly twice as many as before.

Authors. Written by Boris Alexeev with assistance from Valery Alexeev, Mo Hendon, and Ted Shifrin.

Sources. Almost all of these problems are directly inspired by Steve Sigur's emails and other writings, which were often themselves inspired by ARML or AIME problems. Problem 7 is from "Mathematical Miniatures" by Svetoslav Savchev and Titu Andreescu.