Sponsored by: UGA Math Department and UGA Math Club

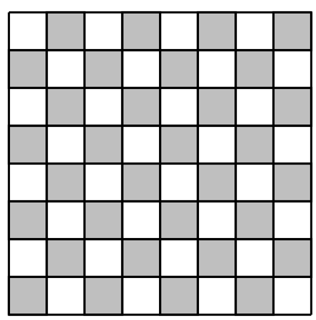
Written test, 25 problems / 90 minutes
November 18, 2006

WITH SOLUTIONS

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

1 Easy Problems

Problem 1. A straight line crosses an 8 by 8 chess board. What is the largest number of squares it can pass through (not just touch the boundary)?



(A) 8 (B)$^\heartsuit$ 15 (C) 16 (D) 17 (E) None of the above

Solution. There are a total of 14 interior lines, 7 vertical and 7 horizontal. Every time the line crosses from one square to another, it crosses at least one of these interior lines. There is also the starting square. Therefore, the line cannot pass through more than $1 + 14 = 15$ squares. The number 15 is possible, by considering a line close to a main diagonal.

Problem 2. The lion always lies on Mondays, Tuesdays, and Wednesdays, but always tells the truth on the other days. The unicorn always lies on Thursdays, Fridays, and Saturdays, but always tells the truth on the other days. If they both announce to you, "I told lies yesterday," what day is it?

(A) Monday (B) Wednesday (C)$^\heartsuit$ Thursday (D) Saturday
(E) None of the above

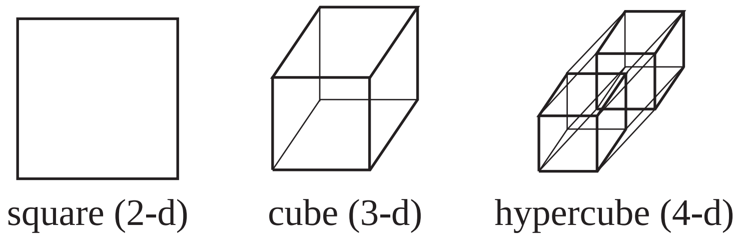
Solution. One of the animals can announce "I told lies yesterday" only if either (a) he is telling the truth today but lied yesterday or (b) he is lying today but told the truth yesterday. For the lion, these two possible days are Monday and Thursday. For the unicorn, they are Thursday and Sunday. Therefore, it is Thursday.

Problem 3. You go to visit the penguins at the zoo. The penguins live in a very cold circular room with walls made of the glass so that you can see them. You notice one especially cute baby penguin standing against the wall. He waddles 15 feet, bumps into the glass, turns 90 degrees, waddles another 36 feet, bumps into glass again, and falls down. What is the diameter of the penguins' habitat?

(A) 25 (B) 26 (C) 27 (D)$^\heartsuit$ 39 (E) None of the above

Solution. Let the penguin's initial point be A, the next point B, and the final point C. Then we are told that A, B, and C lie on a circle and that $\angle ABC = 90°$. Thus $AC = \sqrt{15^2 + 36^2} = 39$ is the diameter of the circle. (Better: Recall the standard Pythagorean triple, 5-12-13.)

Problem 4. An ant is hopping from vertex to vertex along the edges of a 4-dimensional hypercube. To how many places can he get in exactly three hops (he can jump back to where he was before)?



(A) 4 (B)$^\heartsuit$ 8 (C) 11 (D) 15 (E) None of the above

Solution. Consider the cube with the vertices (a_1, a_2, a_3, a_4) with a_i all 0 or 1. Suppose the ant starts at the vertex $(0, 0, 0, 0)$. Then after exactly three hops he will be either at a vertex with exactly three 1's or exactly one 1's. There are 4 vertices of each kind.

Problem 5. A king rules over a vast subset of the Cartesian plane, including a river which occupies all points (x, y) such that $0 < y < 1$. The king wishes to build a road from his castle at $(-2, -4)$ to the queen's castle at $(10, 3)$ that will include a bridge (an interval of length 1 crossing the river at a constant x-coordinate x_0). What choice of x_0 minimizes the length of the road?

(A) 4 (B)$^\heartsuit$ 6 (C) 8 (D) 9 (E) None of the above

Solution. Collapse the river. (That is, move all points (x, y) with $y \geq 1$ to $(x, y - 1)$ and forget about the points that used to be part of the river.) Then the shortest path joining the castles is a straight line. This line will intersect the x-axis at $x_0 = -2 + \frac{2}{3}(12) = 6$.

Problem 6. How many positive integers divide 15! (fifteen factorial)?

(A) 2048 (B) 3168 (C)$^\heartsuit$ 4032 (D) 4096 (E) None of the above

Solution. 15! is divisible by $\lfloor 15/2 \rfloor + \lfloor 15/4 \rfloor + \lfloor 15/8 \rfloor = 11$ powers of 2, and

$\lfloor 15/3 \rfloor + \lfloor 15/9 \rfloor = 6$ powers of 3, etc. In other words

$$15! = 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$$

So it has $(11+1)(6+1)(3+1)(2+1)(1+1)(1+1) = 4032$ divisors.

Problem 7. How many different rectangles whose length and width are both integers have their area equal to 3 times their perimeter?

(A) 1 (B) 2 (C) 4 (D)$^\heartsuit$ 5 (E) infinitely many

Solution. Let a and b denote the length and width. Then we must have $ab = 6(a+b)$, so $(a-6)(b-6) = 36$. Since there are 5 distinct ways of factoring 36 ($36 = 1 \cdot 36 = 2 \cdot 18 = 3 \cdot 12 = 4 \cdot 9 = 6 \cdot 6$), there are 5 different rectangles. An alternative solution can be obtained by writing $b = \dfrac{6a}{a-6}$ and thinking about divisibility of integers.

Problem 8. What is the largest integer n that cannot be represented as $8a + 15b$ with a and b nonnegative integers?

(A) 89 (B)$^\heartsuit$ 97 (C) 117 (D) 119 (E) None of the above

Solution. $98 = 6 \cdot 15 + 8$, $99 = 5 \cdot 15 + 3 \cdot 8$, $100 = 4 \cdot 15 + 5 \cdot 8$, $101 = 3 \cdot 15 + 7 \cdot 8$, $102 = 2 \cdot 15 + 9 \cdot 8$, $103 = 1 \cdot 15 + 11 \cdot 8$, $104 = 13 \cdot 8$, $105 = 7 \cdot 15$. That gives 8 consecutive numbers. We can now get any larger number by adding a multiple of 8 to one of 98 through 105. If 97 could be expressed as $8a + 15b$, note that b would have to be odd, but $97 - 15$, $97 - 45$, and $97 - 75$ are not multiples of 8.

For the general case of relatively prime positive $m, n > 1$, the largest number that cannot be written in the form $ma + nb$ with $a, b \geq 0$ is $mn - m - n$.

Problem 9. Find

$$\frac{1}{1} + \frac{1}{2} + \frac{2}{2} + \frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \cdots + \frac{9}{10} + \frac{10}{10}$$

(A) $\frac{55}{2}$ (B) 30 (C)$^{\heartsuit}$ $\frac{65}{2}$ (D) 37 (E) None of the above

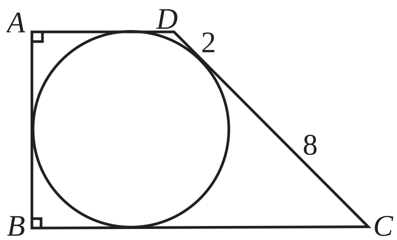
Solution.

$$\frac{1}{n} + \cdots + \frac{n}{n} = \frac{n(n+1)/2}{n} = \frac{n+1}{2}$$

and

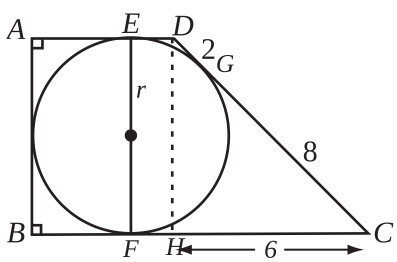
$$\frac{1+1}{2} + \cdots + \frac{10+1}{2} = \frac{1+2+\cdots+10}{2} + \frac{10}{2} = \frac{10 \cdot 11/2}{2} + \frac{10}{2} = \frac{65}{2}\,.$$

Problem 10. A circle is inscribed in a trapezoid $ABCD$. If $\angle DAB = \angle ABC = 90°$ and the circle's point of tangency divides line segment $\overline{CD}$ into segments of length 2 and 8, what is the perimeter of the trapezoid?



(A) 32 (B)$^{\heartsuit}$ 36 (C) 40 (D) 48 (E) None of the above

Solution. Joining the other points of tangency (E and F), we get the following picture:



Note that $\overline{EF}$ is a diameter of the circle, and so the perimeter is $4r + 2 \cdot 2 + 2 \cdot 8 = 4r + 20$. But $EF = DH$ and $HC = 8 - 2 = 6$, so, recognizing a 3-4-5 triangle, we have $2r = DH = 8$. So $r = 4$ and the perimeter is 36.

2 Medium Problems

Problem 11. Ten people each have exactly one unique secret. When one of them calls another, the caller tells every secret he knows, but learns nothing from the person he calls. How many phone calls will be needed in order for each person to know all ten secrets?

(A)$^\heartsuit$ 18 (B) 19 (C) 20 (D) 21 (E) None of the above

Solution. We will do this problem for n people and then will take $n = 10$.

Everyone must receive a call after $n-2$ calls have been made, inasmuch as calls up to and including the $(n-2)$th can communicate in total at most $n-2$ secrets, and each person needs to hear $n-1$ secrets. Thus, the minimum number of calls is $2n-2$. This can be achieved as follows. Let every person call a designated "custodian of the secrets," who then calls everyone back.

Problem 12. Three circles of radius 1 are tangent to one another. What is the radius of the smallest circle that contains all of them?

(A) 2 (B)$^\heartsuit$ $1+\frac{2\sqrt{3}}{3}$ (C) $1+\sqrt{2}$ (D) $\pi/2$ (E) None of the above

Solution. The centers of the 3 circles form an equilateral triangle of side 2. The center of the fourth circle must obviously be at the centroid of this triangle. Since the centroid lies 2/3 the way down each median, the distance to a vertex is $\frac{2\sqrt{3}}{3}$, so the radius of the fourth circle is $1+\frac{2\sqrt{3}}{3}$.

There is a more general *Kissing Theorem* due to Descartes, which says that for four mutually tangent circles the radii must satisfy the following equation:

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2 = 2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)$$

Plugging in $a=b=c=1$ and solving for d gives $d = -\left(1+\frac{2}{3}\sqrt{3}\right)$, and the minus sign appears because the last circle touches the first three "from the other direction." (The other solution corresponds to the small circle internally tangent to all three circles.) The theorem remains true if one of the circles is replaced by a line (a "circle" of radius ∞).

Problem 13. The square of an integer has tens digit 7. What is the units digit?

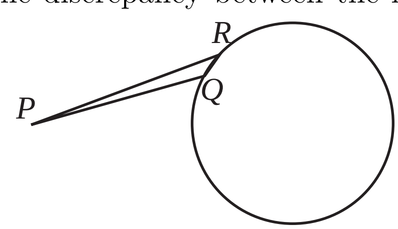
(A) 0 (B) 2 (C) 4 (D)$^\heartsuit$ 6 (E) 8

Solution. Suppose the original integer has tens digit a and units digit b. Since $(10a+b)^2 = 100a^2 + 20ab + b^2$, the carry from b^2 must be odd (since 7 is odd), so $b = 4$ or 6. In either case, the units digit of the square is 6. An example is $24^2 = 576$.

Problem 14. What is the length of the shortest path from the point $(-2, 0)$ to $(2, 0)$ that avoids the interior of the circle of radius 1 centered at the origin?

(A)$^\heartsuit$ $2\sqrt{3} + \pi/3$ (B) $2\sqrt{5}$ (C) $2 + \pi$ (D) $2\sqrt{5 - 2\sqrt{2}} + \pi/2$
(E) None of the above

Solution. The shortest path is obtained by going from $(-2, 0)$ to the circle along a tangent to the circle, walking around the circle to $(0, 1)$, and then continuing symmetrically. By the Pythagorean theorem, the length of the tangent is $\sqrt{2^2 - 1^2} = \sqrt{3}$. The arc from the point of tangency to $(0, 1)$ is $\pi/6$. Here's a plausibility argument that the line from the point should be tangent to the circle. If it were not, we could get a shorter path by moving a bit further along the circle: In $\triangle PQR$, we know that $PR < PQ + QR$ (and if R is very close to Q on the circle, the discrepancy between the length of



$\overline{QR}$ and that of arc QR is very small.)

Problem 15. How many paths are there starting from $(0, 0, 0)$ and ending at $(2, 2, 2)$ where each step consists of increasing exactly one of the three coordinates by 1?

(A)$^\heartsuit$ 90 (B) 120 (C) 180 (D) 729 (E) None of the above

Solution. Write a letter x (respectively y, z) every time the x-coordinate is increased (resp. the y-coordinate, the z-coordinate). Then we can write every path in the form $xxyyzz$, $xxyzyz$, etc: a combination of 6 letters, with exactly two each of x, y, and z. The answer is

$$\binom{6}{2} \cdot \binom{4}{2} = \frac{6!}{2!\,4!} \cdot \frac{4!}{2!\,2!} = \frac{6!}{2!\,2!\,2!} = \frac{720}{8} = 90.$$

Indeed, there are $\binom{6}{2}$ ways to choose the places for x and then $\binom{4}{2}$ ways to choose the places for y.

Perhaps a better way of looking at this is as the *multinomial coefficient* of 6 choose 2, 2, and 2, written variously as $\binom{6}{2\ 2\ 2}$ or $(2; 2; 2)$ and defined as

$$\binom{a+b+c}{a\ b\ c} = \frac{(a+b+c)!}{a!\,b!\,c!}.$$

Problem 16. In how many ways can we arrange 7 (identical) white balls and 5 (identical) black balls in a row so that there is at least one white ball between any two black balls?

(A) 35 (B) 48 (C)$^\heartsuit$ 56 (D) 120 (E) None of the above

Solution. We have BWBWBWBWB and 3 spare W which can go anywhere. There are 6 different places to put each W (at either end or with one of the existing W). If we put each W in a different place then there are $6 \cdot 5 \cdot 4/3!$ $= 20$ possibilities. If we put two in one place and one in another, there are $6 \cdot 5 = 30$ possibilities. If we put them all in the same place, there are 6 possibilities. Total 56. As a check, put down the 7 W first. Then there are 8 places to put the five B, and each must go in a different place. Hence $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4/5! = 56$.

Problem 17. Find the length of the longest possible geometric progression in $\{100, 101, 102, \ldots, 1000\}$.

(A) 5 (B)$^\heartsuit$ 6 (C) 7 (D) 8 (E) None of the above

Solution. 128, 192, 288, 432, 648, 972 has six terms and ratio 3/2. We show it is the best possible.

The ratio must be rational. Suppose it is h/k (in lowest terms). If there are more than 6 terms, then the first term must be divisible by k^6 and the last term must be divisible by h^6. Note that we must have $h \leq 3$, since $4^6 > 1000$. Since $h > k$, we must therefore have $k < 3$. On the other hand, we cannot have $k = 1$, because then the last term is at least h^6 times the first term, which is impossible. So $k = 2$ and $h = 3$. So the first term must be a multiple of 64. The smallest such is 128. But then we only get 6 terms.

Problem 18. Suppose a, b, c, and n are positive integers. How many solutions of $n! = a! + b! + c!$ are there? (The different ways of permuting a, b, and c count as the same solution.)

(A) 0 (B)$^\heartsuit$ 1 (C) 2 (D) 3 (E) None of the above

Solution. We must have $n > a, b, c$ or else $n! < a! + b! + c!$. But if $n > a, b, c$ and $n > 3$, then $n! > 3(n-1)! \geq a! + b! + c!$. So we must have $n = 1$, 2, or 3. It is easy to check that this gives $3! = 2! + 2! + 2!$ as the only solution.

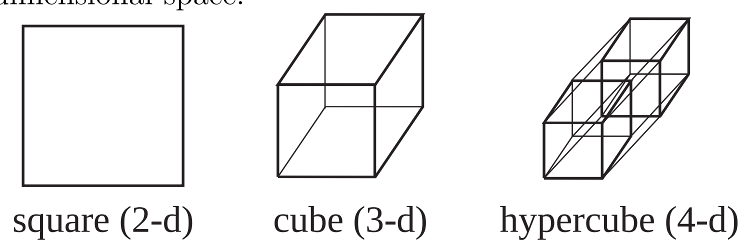
Problem 19. What is the probability of getting no successive heads when one flips a fair coin 7 times in a row?

(A) 27/128 (B) 1/4 (C) 33/128 (D)$^\heartsuit$ 17/64
(E) None of the above

Solution. Let a_n be the number of heads/tails sequences of length n having no successive heads (H). Let's call such sequences "good." Then $a_1 = 2$ (H or T) and $a_2 = 3$ (HT, TH, or TT). Every good sequence of length n is either T followed by a good sequence of length $n-1$ or HT followed by a good sequence of length $n-2$. Thus, $a_n = a_{n-1} + a_{n-2}$. So (notice the appearance of the Fibonnaci sequence) $a_3 = 5$, $a_4 = 8$, $a_5 = 13$, $a_6 = 21$, and

$a_7 = 34$. So the probability is $34/2^7 = 17/2^6 = 17/64$.

Problem 20. A major diagonal in a 4-dimensional hypercube goes from a vertex to the opposite vertex farthest from it. What is the smallest nonzero angle between two major diagonals? The angle is measured in the 4-dimensional space.



(A) 30° (B) 45° (C)$^{\heartsuit}$ 60° (D) 90° (E) None of the above

Solution. (This solution requires some knowledge of dot products.)

Consider the cube with the vertices (a_1, a_2, a_3, a_4) with all $a_i = 0$ or 1. Then the major diagonals are all of the form

$$\vec{u} = (\pm 1, \pm 1, \pm 1, \pm 1).$$

If $\vec{u} = (u_1, u_2, u_3, u_4)$ and $\vec{v} = (v_1, v_2, v_3, v_4)$ are two diagonals and θ is the angle between them then

$$\cos\theta = \frac{(u, v)}{|u|\ |v|} = \frac{u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4}{2 \cdot 2}$$

since

$$|u| = \sqrt{(\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2} = 2.$$

The possibilities for the numerator $\pm 1 \pm 1 \pm 1 \pm 1$ are 0, ± 2 and ± 4. This gives $\cos\theta = 0$, $\pm 1/2$ or ± 1, hence $\theta = 90°$, $60°$, $120°$, $0°$ and $180°$.

3 Hard Problems

Problem 21. Bob has a bag with two coins, one of them fair and one of them double-headed. He randomly chooses one of the coins from the bag and flips it three times. Given that it landed heads each time, what is the probability that it is the fair coin?

(A) $\frac{1}{16}$ (B)$^{\heartsuit}$ $\frac{1}{9}$ (C) $\frac{1}{8}$ (D) $\frac{1}{4}$ (E) None of the above

Solution. Write $P(A \mid B)$ for the probability that A is true given that B is true. (This is standard notation.) Then Bayes' Theorem says that

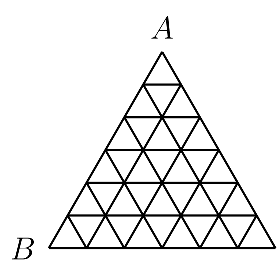
$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}.$$

If we let A be "the chosen coin is fair" and B be "the coin comes up heads all three times," we may compute

$$P(A \mid B) = \frac{\left(\frac{1}{8}\right) \cdot \left(\frac{1}{2}\right)}{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{8}} = \frac{\frac{1}{16}}{\frac{9}{16}} = \frac{1}{9}.$$

To save some computation, one may interpret Bayes' Theorem as saying that "$P(A \mid B)$ is proportional to $P(B \mid A)$" in the sense that the probability that the coin comes up heads three times if it was fair is $1/8$, while if it was double-headed is 1, and whence the probability that it is fair is $\frac{1/8}{1/8+1} = \frac{1}{9}$.

Problem 22. How many paths are there from A to B which do not pass through any vertex twice and which move only downwards or sideways, never up?



(A) 1024 (B) 2048 (C) 23040 (D)$^{\heartsuit}$ 46080
(E) None of the above

Solution. If the large triangle has side n and top vertex A, let a_n be the number of ways of getting from A to the bottom row (the path stops as soon as it gets to the bottom row). Note that a_n is also the number of ways of getting from A to the bottom left-hand vertex, because having reached the bottom row there is only one allowed path to the bottom left-hand vertex. Obviously, $a_1 = 2$. (Also $a_0 = 1$.)

Now consider a_{n+1}. There are a_n ways to get to the bottom row but one. Now there is just one way of getting to any vertex on the bottom row and there are then 2 ways to get to the row below. So $a_{n+1} = 2(n+1)a_n$. Hence $a_6 = 2^5 6! a_1 = 2^6 6!$.

Problem 23. What is the remainder left upon dividing the binomial coefficient $\binom{2006}{1118}$ by 13?

(A) 0, 1, or 2 (B) 3, 4, or 5 (C) 6, 7, or 8 (D) 9 or 10
(E)$^\heartsuit$ 11 or 12

Solution. Let $(a_n \cdots a_0)_p$ represent a number written in base p, with the digits $0 \le a_i < p$. Then Lucas's theorem says that if p is prime,

$$\binom{(a_n \cdots a_0)_p}{(b_n \cdots b_0)_p} \equiv \binom{(a_n)_p}{(b_n)_p} \cdots \binom{(a_0)_p}{(b_0)_p} \pmod{p}.$$

In our case 2006 is $(11, 11, 4)_{13}$ and 1118 is $(6, 8, 0)_{13}$ in base 13. We may then compute

$$\binom{2006}{1118} \equiv \binom{11}{6} \cdot \binom{11}{8} \cdot \binom{4}{0} \equiv 7 \cdot 9 \cdot 1 \equiv 11 \pmod{13}.$$

Problem 24. Suppose a sequence $a_0, a_1, \ldots$ is chosen randomly, with each a_i independently either $+1$ with probability $3/4$ or -1 with probability $1/4$. What is the probability that for some $n \ge 0$, we have $\sum_{i=0}^{n} a_i < 0$?

(A)$^\heartsuit$ 1/3 (B) 1/2 (C) 2/3 (D) 1 (E) None of the above

Solution. Generalize this problem so that $+1$ appears with probability $p \ge 1/2$. Let C_n be the number of sequences of ± 1 of length $2n$ such that all partial sums satisfy $\sum_{i=0}^{k} a_i \geq 0$. These are commonly known as the *Catalan numbers*, but this solution will not assume prior knowledge of them. Then the probability we are looking for is $\sum_n C_n p^n (1-p)^{n+1} = (1-p) \cdot \sum_n C_n (p \cdot (1-p))^n = (1-p) \cdot f(p \cdot (1-p))$ for $f(x) = \sum_n C_n x^n$. Indeed, there are C_n sequences of length $2n+1$ such that $\sum_{i=0}^{k} a_i \geq 0$ for $k \leq 2n$ but are -1 for $k = 2n+1$; in other words, there is a one-to-one correspondence between Catalan paths and sequences that have some partial sum negative.

Thus we must find the Catalan numbers' *generating function* $f(x) = \sum_n C_n x^n$. To do this, we note that the Catalan numbers satisfy the recurrence $C_0 = 1$ and $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. (Take a Catalan sequence of length $2(n+1)$ and let $k > 0$ be the first partial sum $\sum_{i=0}^{2k} a_i$ that equals 0, with perhaps $k = n+1$. Then the Catalan sequence looks like $+1$, a Catalan sequence of length $k-1$, -1, and a Catalan sequence of length $n+1-k$.) It follows that the generating function f satisfies the function equation $f(x) - 1 = x \cdot f(x)^2$. Thus by the quadratic formula, $f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$; one can check that the correct sign is $-$.

Plugging this into our relation earlier, we obtain $(1-p) \cdot f(p \cdot (1-p)) = (1-p) \cdot \frac{1-\sqrt{1-4p(1-p)}}{2p(1-p)} = (1-p) \cdot \frac{1-(2p-1)}{2p(1-p)} = \frac{1-p}{p}$. Plugging in $p = 3/4$, we obtain $1/3$. (Note that if $p \leq 1/2$, the sign of the square root is the opposite and one obtains 1.)

Problem 25. Boris wanders along a river and gets lost, two miles from home in one direction and three miles from his friend's house in the other, but he doesn't remember which is which. Every hour, Boris randomly and with equal probability chooses a direction and walks a mile in that direction (thus perhaps repeatedly walking along the same stretches of the river). He stops when he reaches one of the two homes. What is the probability that it is his own?

(A) 1/2 (B) 2/3 (C) 3/4 (D) 4/5 (E)$^\heartsuit$ None of the above

Answer. 3/5

Solution. Let Boris's location at time t be a_t (measured in miles), with that $a_0 = 0$, his house at -2, and his friend's house at $+3$. Then the quantity a_t

is a *martingale*; i.e., the expectation $E(a_t \mid a_s \text{ given}) = a_s$ if $s \le t$.

Suppose further that the first time $a_t \in \{2, -3\}$ is for $t = T$. Then $a_{\min(t,T)}$ is also a martingale. Suppose the probability that Boris reaches his own home is p. We may then solve

$$E(a_T \mid a_0 = 0) = p \cdot (-2) + (1 - p) \cdot 3 = 0$$

to obtain $p = 3/5$.

Second solution. One can show that the probability $f(x, y)$ of reaching $-x$ before y is $y/(x + y)$. Indeed, $f(x, 0) = 0$, $f(0, y) = 1$ and

$$\begin{aligned} f(x, y) &= \frac{y}{x + y} = \frac{1}{2} \cdot \frac{y + 1}{x + y} + \frac{1}{2} \cdot \frac{y - 1}{x + y} \\ &= \frac{1}{2} f(x - 1, y + 1) + \frac{1}{2} f(x + 1, y - 1). \end{aligned}$$

Third, more elementary solution. So, for when Boris is 2/5 away from his house and 3/5 from a friend's, let us put that on the number line where we will designate Boris' home as $x = 0$ and his friend's house at $x = 5$. Let $P(a = i)$ denote the probability that Boris will get home before he gets to his friend's house given that he is currently at $x = i$. Boris starts at $x = 2$. So, $P(\text{Boris gets home}) = P(a = 2)$.

So, we take the first two iterations. There is a 1/4 chance he goes to $x = 0$, 1/2 chance he goes to $x = 2$, and a 1/4 chance he ends up at $x = 4$. Then we reevaluate the probability.

$$P(a = 2) = P(a = 0)/4 + P(a = 2)/2 + P(a = 4)/4.$$

$P(a = 0) = 1$, obviously. Strategy: we need to find $P(a = 4)$ in terms of $P(a = 2)$.

Examine $P(a = 4)$ using one iteration of Boris' random walking.

$$P(a = 4) = P(a = 3)/2 + P(a = 5)$$

If he moves in the + direction, he ends up at his friend's, so $P(a = 5) = 0$.

$$P(a = 3) = P(a = 2)/2 + P(a = 4)/2$$

So,

$$P(a=4) = P(a=2)/4 + P(a=4)/4.$$

Hence, $P(a=4) = P(a=2)/3$.

Thus,

$$\begin{aligned}
P(a=2) &= P(a=0)/4 + P(a=2)/2 + P(a=4)/4 \\
P(a=2) &= 1/4 + P(a=2)/2 + P(a=2)/12 \\
P(a=2) &= 3/5
\end{aligned}$$

Authors. Problems by Boris and Valery Alexeev, with contributions by Meredith Perrie, Tyler Kelly, Ted Shifrin and Mo Hendon. Some problems taken from olympiad problems from around the world collected by John Scholes at `http://www.kalva.demon.co.uk`, as well as from a newspaper comic.