University of Georgia High School Math Tournament

1785

2004

Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

WITH SOLUTIONS

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

1 Easy Problems

Problem 1. What is

$$\lceil (6-\sqrt{6})^6 \rceil \ ?$$

(here, $\lceil x \rceil$ denotes the *ceiling* function, i.e. the smallest integer n with $n \geq x$).

(A) 0 (B) π (C) 6 (D)$^\heartsuit$ 2004 (E) 5^6

Solution. This is just a cool fact for a warm-up. See http://mathpuzzle.com/2004.txt and the sequence A091146 at the Encyclopedia of Integer Sequences http://www.research.att.com/~njas/sequences/ for more information.

Problem 2. Find

$$\frac{6!4!2!}{5!3!1!}$$

(A) 6 (B) 12 (C) 24 (D)$^\heartsuit$ 48 (E) None of the above

Solution. One has

$$\frac{n!}{(n-1)!} = \frac{1 \cdot 2 \cdot 3 \cdot (n-1) \cdot n}{1 \cdot 2 \cdot 3 \cdot (n-1)} = n$$

Therefore,

$$\frac{6!4!2!}{5!3!1!} = \frac{6!}{5!} \cdot \frac{4!}{3!} \cdot \frac{2!}{1!} = 6 \cdot 4 \cdot 2 = 48$$

Problem 3. In what base b, with $b > 7$, is $3 \cdot 7 = 18$?

(A) 8 (B) 10 (C) 12 (D) 14 (E)$^\heartsuit$ None of the above

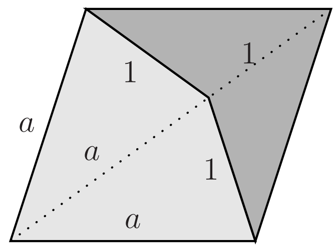
Solution. 18 in base b means $1 \cdot b + 8$, so $b + 8 = 21$ and $b = 13$. Hence, the right answer is “None of the above”.

Problem 4. What will be the next year when the calendar will be exactly the same as in 2004 (i.e. all the days, January 1 through December 31, will fall on the same days of the week)?

(A) 2005 (B) 2008 (C)$^\heartsuit$ 2032 (D) 2040 (E) Never

Solution. It will have to be a leap year, i.e. 2008 or 2012 etc. In a non-leap year, the calendar is shifted by $365 \equiv 1 \pmod 7$ days, and in a leap year by $366 \equiv 2 \pmod 7$ days. Therefore, by the next leap year, the calendar will shift by $1+1+1+2 \equiv 5$ days. It will be shifted by $0 \pmod 7$ days in $7 \cdot 4 = 28$ years, and the year will be $2004 + 28 = 2032$.

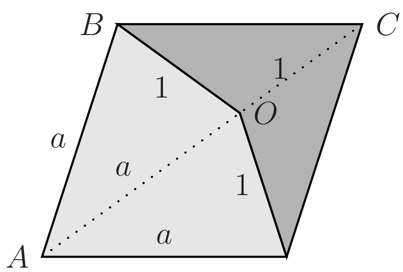
Problem 5. In the picture below, what is a? (The large figure is a rhombus.)

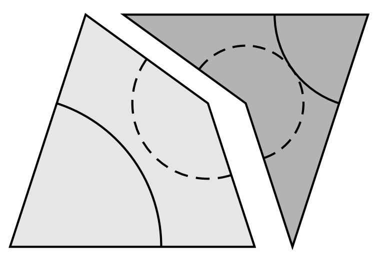


(The two tiles pictured are called Penrose tiles, "kite" and "bat".)

(A) $\sqrt{5}-1$ (B) 1 (C) 2 (D)$^{\heartsuit}$ $\dfrac{1+\sqrt{5}}{2}$ (E) None of the above

Solution.



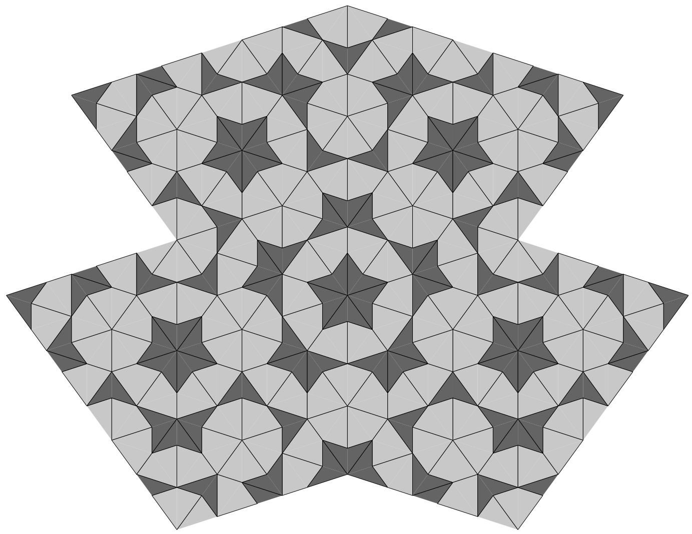


The triangles BOC and ABC are similar. This gives

$$\frac{a}{1}=\frac{BC}{CO}=\frac{AC}{AB}=\frac{a+1}{a}$$

Therefore, a satisfies the quadratic equation $a^2=a+1$. Solving, we obtain $a=(1+\sqrt{5})/2$. This number has been known since antiquity and is usually called the **golden ratio.**

These tiles were discovered in 1970s by an Oxford mathematician Penrose. He proved the following amazing fact. Mark the tiles as shown in the picture on the right and try to tile the plane with them so that the markings match. (In particular, the kite and the bat never form a rhombus!) Then one obtains infinitely many tilings (see an illustration below for one example), all of them highly symmetric but none of them periodic! Such tilings are called aperiodic.



Sir Roger Penrose, Rouse Ball Professor of Mathematics at Oxford will visit the UGA Department of Mathematics next spring to give the Cantrell Lectures. The first lecture is going to be accessible to a wide audience, including high school students (in fact, we assume that it will be on Penrose tilings!) Watch **http://www.math.uga.edu/seminars_conferences/cantrell.html** for further announcements and plan to attend!

Problem 6. In a 4×4 magic square, with numbers 1 through 16, what is the sum of the numbers in any row or column? (In a magic square, the numbers in each row, column, and diagonal have the same sum.)

(A) 16 (B) 17 (C) 32 (D) 36 (E)$^\heartsuit$ None of the above

Solution.

$$1 + 2 + \cdots + 16 = \frac{16 \cdot 17}{2}$$

There are 4 rows, and the sum in each row is the same. Therefore, for every row it is

$$\frac{16 \cdot 17}{2 \cdot 4} = 2 \cdot 17 = 34$$

Problem 7. There are four cards, two red and two black. Two cards are chosen at random. What is the probability that they have the same color?

(A) 1/4 (B)$^\heartsuit$ 1/3 (C) 1/2 (D) 2/3 (E) None of the above

Solution. Pick the first card arbitrarily. Of the remaining three cards, only one has the same color. So the probability that the two cards have the same color is 1/3.

Problem 8. Call a positive integer *evil* if it is 666 larger than the sum of its digits. How many evil numbers are there?

(A) 0 (B) 9 (C)$^\heartsuit$ 10 (D) 18 (E) 666

Solution. If the number is $a_n a_{n-1} \dots a_1 a_0$ then our condition says

$$10^n a_n + 10^{n-1} a_{n-1} + \cdots + 10 a_1 + a_0 = a_n + a_{n-1} + \cdots + a_1 + a_0 + 666$$

In other words,

$$9 \dots 9 a_n + 9 \dots 9 a_{n-1} + \cdots + 9 a_1 = 666$$

So, for a 2-digit number we get $9a_1 = 666$ – no solutions, a_1 is too small, and for a 4- or more-digit number we get $999a_3 + \cdots = 666$ – no solutions, the numbers are too large.

The only possibility is a 3-digit number, and then

$$99a_2 + 9a_1 = 666, \qquad \text{so} \qquad 11a_2 + a_1 = 74.$$

The only solutions are: $a_2 = 6$, $a_1 = 8$, and a_0 is arbitrary. Hence, there are 10 solutions.

Problem 9. In a Rubik's cube (consisting of $3 \times 3 \times 3 = 27$ smaller cubes) how many diagonals of all kinds there are?

Here, a *diagonal* is defined to be a straight line consisting of 3 distinct cells. For example, in a 3×3 square there are $3 + 3 + 2 = 8$ diagonals, 3 horizontal, 3 vertical and 2 going from one corner to another.

(A) 27 (B) 35 (C)$^\heartsuit$ 49 (D) 76 (E) None of the above

Solution. There are $9 + 9 + 9 = 27$ short diagonals, $2 \cdot 3 \cdot 3 = 18$ longer and 4 longest. The total is $27 + 18 + 4 = 49$.

Problem 10. You are in charge of designing a new system of coins. You are allowed to design coins with any integral value but executive order demands that any integral amount from 1 to 99 cents be obtainable without using any type of coins twice. What is the smallest number of types of coins that you must have?

(A) 5 (B) 6 (C)$^\heartsuit$ 7 (D) 8 (E) More that 8

Solution. 7 types of coins suffice: for example, coins worth 1, 2, 4, 8, 16, 32 and 64. Fewer coins cannot work since with n coins there is at most 2^n combinations where each coin is used 0 or 1 times; and $2^6 = 64 < 100$.

Problem 11. Three circles of radius 1 are centered at the vertices of the 3–4–5 triangle. (When we say "circle" we include the boundary but not the interior.) A point on one of the circles is called *invisible* if it cannot be seen from one of the other circles (one cannot see through the circles). What is the total length of the set of invisible points?

(A) 3 (B) π (C) 5 (D)$^\heartsuit$ 2π (E) None of the above

Solution. No solution offered. Try to find it yourself!

Problem 12. Find

$$\sum_{j=0}^{5} \binom{5}{j} (-2)^j$$

(A)$^\heartsuit$ -1 (B) 0 (C) 1 (D) 243 (E) None of the above

Solution. Using the binomial formula,

$$\sum_{j=0}^{5} \binom{5}{j} (-2)^j = (1-2)^5 = (-1)^5 = -1$$

Problem 13. Find the minimum value of the following function:

$$f(x) = (x-5)^2 + (x-7)^2 - (x-4)^2 - (x-8)^2 + (x-3)^2 + (x-9)^2$$

(A) 8 (B) 10 (C)$^\heartsuit$ 12 (D) 14 (E) None of the above

Solution. The function $f(x)$ is quadratic and the coefficient of x^2 is $1+1-1-1+1+1 = 2 > 0$, so the graph of $f(x)$ is an upward-pointing parabola, and the minimum is attained at its vertex. The function is also *symmetric* about $x = 6$, so the vertex must be at $x = 6$. We easily compute

$$f(6) = 1+1-4-4+9+9 = 2(1+9-4) = 12$$

2 Medium Problems

Problem 14. How many different words can you form from the letters ABCD, where a word is a sequence of one to four letters, using every letter at most once (for example, words DC and CADB)?

(A) 24 (B) 48 (C)$^\heartsuit$ 64 (D) 72 (E) None of the above

Solution.

$$4 \cdot (1 + 3 \cdot (1 + 2 \cdot (1 + 1))) = 4 \cdot (1 + 3 \cdot (1 + 4)) = 4 \cdot (1 + 15) = 64.$$

Problem 15. The centers of the faces of a regular tetrahedron are connected to form a smaller tetrahedron. What is the ratio of the volumes of the bigger and smaller tetrahedra?

(A) 5 (B) 8 (C) 9 (D)$^\heartsuit$ 27 (E) 64

Solution. Let a, b, c, d be the vectors from the center of the tetrahedron to its four vertices. We have $a+b+c+d = 0$. Then the vertices from the center to the vertices of the smaller tetrahedron are

$$\frac{a+b+c}{3} = \frac{a+b+c}{3} - \frac{a+b+c+d}{3} = -\frac{d}{3}$$

and, similarly, $-a/3$, $-b/3$, $-c/3$. So, the smaller tetrahedron is 3 times smaller, and its volume is $3^3 = 27$ times smaller.

[Remember that the centroid of a triangle is $\frac{2}{3}$ down each diagonal.]

Problem 16. In a certain fictional country families stop having children precisely when they produce both a boy and a girl (at least one of each). Assuming boys and girls are equally likely, how many children does an average family have?

(A)$^\heartsuit$ 3 (B) 3.5 (C) 4 (D) 5 (E) None of the above

Solution. Let us say the first child is a boy. After that point, the parents continue on having children until they have a girl. So, the expected number of girls *after the first child* is 1. But since boys and girls are equally likely, the expected number of boys after the first child is also 1. So, the answer is $1 + (1 + 1) = 3$.

Second (much harder) solution. The expected value for the number of children *after the first one (which we assume was a boy)* is computed by the following power series, with $x = 1/2$:

$$f(x) = x + 2x^2 + 3x^3 + \cdots = x(1 + 2x + 3x^2 + \dots) =$$
$$x(1 + x + x^2 + x^3 + \dots)' = x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}$$

Indeed, a girl will be the first child with probability x, and the second child with possibility x^2, the third with probability x^3 etc. Plugging $x = 1/2$ in this formula gives $f(1/2) = 2$, and so the answer is 1+2=3.

Problem 17. A person is standing on a rectangular grid. He is allowed to move one step south, north, east or west. After 4 moves he is supposed to get back to the starting point. How many possibilities are there for the sequence of these 4 moves?

(A) 20 (B) 24 (C) 30 (D)$^\heartsuit$ 36 (E) None of the above

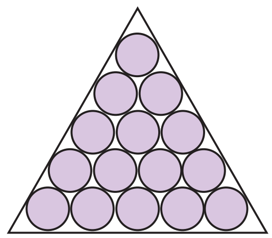
Solution. There must be 2 moves in the NW directions (north or west) and 2 in the opposite, SE directions. That gives

$$\binom{4}{2} = 6$$

possibilities. Similarly, there are 2 moves in the NE directions and 2 in SW directions, again 6 possibilities.

For every move, if we know which groups: NW/SE and NE/SW it belongs, then its direction is computed uniquely. So, there are $6 \times 6 = 36$ possibilities in all.

Problem 18. An equilateral triangle is filled with n rows of (shaded) congruent circles. (The case $n = 5$ is pictured below.) As the number n goes to infinity, the fraction of the triangle that is shaded approaches

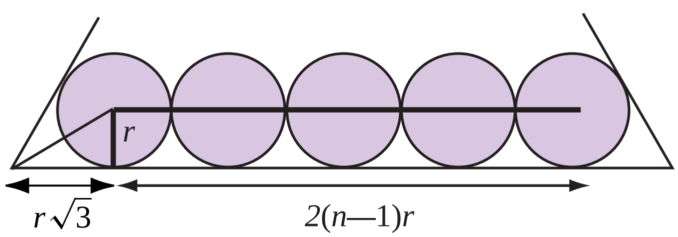


(A) $3/4$ (B)$^{\heartsuit}$ $\dfrac{\pi}{2\sqrt{3}}$ (C) $3/\pi$ (D) 1 (E) None of the above

Solution. Suppose the equilateral triangle has sidelength 1. If the radius of the circles is r, then we must have

$$2(n-1)r + 2r\sqrt{3} = 1,$$

and so $r = \dfrac{1}{2(n+\sqrt{3}-1)}$.



On the other hand there are

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

circles, so their total area is

$$\frac{n(n+1)}{2}\pi r^2 = \frac{\pi}{8}\frac{n(n+1)}{(n+\sqrt{3}-1)^2} \to \frac{\pi}{8} \quad \text{as } n \to \infty.$$

Thus, the ratio of that area to the area of the triangle is

$$\frac{\frac{\pi}{8}}{\frac{\sqrt{3}}{4}} = \frac{\pi}{2\sqrt{3}} \approx 0.91.$$

Problem 19. Let $f(x) = \{5x/2\}$, the fractional part of $5x/2$. How many solutions in the interval $0 \le x < 1$ does the following equation have?

$$f(f(f(x))) = 0$$

(A) 12 (B) 18 (C) 27 (D) 79 (E)$^\heartsuit$ None of the above

Solution. By looking at the graph of $f(x)$, we see that the equation $\{5x/2\} = a$ has 3 solutions $(2/5)a$, $(2/5)(1+a)$, $(2/5)(2+a)$ if $0 \le a < 1/2$ and 2 solutions $(2/5)a$ and $(2/5)(1+a)$ if $1/2 \le a < 1$. Now,

$$\begin{aligned}
&\{5x/2\} = 0 \Rightarrow x = 0,\ 2/5 \text{ or } 4/5 \\
&\{5x/2\} = 2/5 \Rightarrow x = (2/5)^2 = 4/25 \text{ or } (7/5)(2/5) = 14/25 \text{ or } (12/5)(2/5) = 24/25 \\
&\{5x/2\} = 4/5 \Rightarrow x = (4/5)(2/5) = 8/25 \text{ or } (9/5)(2/5) = 18/25
\end{aligned}$$

So, $f(f(x)) = 0$ has solutions

$$0,\ 2/5,\ 4/5,\ 4/25,\ 14/25,\ 24/25,\ 8/25,\ 18/25$$

and $f(f(f(x))) = 0$ has

$$3+3+2+3+2+2+3+2 = 20 \text{ solutions}$$

3 Hard Problems

Problem 20. Arrange e^π, $3^{\sqrt{5}}$ and π^e in increasing order.

(A) $e^\pi < 3^{\sqrt{5}} < \pi^e$ (B)$^\heartsuit$ $3^{\sqrt{5}} < \pi^e < e^\pi$ (C) $\pi^e < 3^{\sqrt{5}} < e^\pi$ (D) $3^{\sqrt{5}} < e^\pi < \pi^e$ (E) $e^\pi < \pi^e < 3^{\sqrt{5}}$

Solution. Taking logarithms on both sides, we see that

$$e^\pi > \pi^e \Leftrightarrow \pi \ln e > e \ln \pi \Leftrightarrow \frac{\ln e}{e} > \frac{\ln \pi}{\pi}$$

Now consider the function $f(x) = (\ln x)/x$. We have

$$f'(x) = \frac{x \cdot 1/x - \ln \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

So, $f'(x) = 0 \Leftrightarrow \ln x = 1$, i.e. $x = e$. For $x > e$, $1 - \ln x < 0$, so the function is decreasing; and for $x < e$, $1 - \ln x > 0$, so the function is increasing. Therefore, $f(x)$ attains a maximum at $x = e$. Hence,

$$\frac{\ln e}{e} > \frac{\ln \pi}{\pi} \quad \text{and} \quad e^\pi > \pi^e$$

In addition, $\pi > 3$ and $e > \sqrt{5}$, obviously, so $\pi^e > 3^{\sqrt{5}}$. So the right answer is

$$e^\pi > \pi^e > 3^{\sqrt{5}}.$$

Problem 21. What is the largest area of an ellipse that can be inscribed in a triangle with sides 3, 4 and 5?

(A) π (B) $\pi\sqrt{2}/2$ (C) $\sqrt{3}\pi/2$ (D)$^\heartsuit$ $2\pi/\sqrt{3}$ (E) None of the above

Solution. This is a problem in affine geometry. Make a linear (affine) transformation mapping our triangle into an equilateral triangle. Inscribed ellipses will map into inscribed ellipses, and the ratios of areas will stay the same.

In an equilateral triangle, the ellipse with the largest area will be clearly the inscribed circle. If the side of the equilateral triangle is 1 (and the area is $\frac{\sqrt{3}}{4}$) then the radius of the inscribed circle is $\frac{1}{3} \cdot \frac{\sqrt{3}}{2}$, and the area is $\frac{\pi}{12}$.

So the ratio of areas of the circle and the triangle is

$$\frac{\pi/12}{\sqrt{3}/4} = \frac{\pi}{3\sqrt{3}}$$

For the 3-4-5 triangle, with the area 6, the area of the largest ellipse will be

$$6\frac{\pi}{3\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

Problem 22. There are 3 light switches turned off. Every minute a person comes by and randomly flips exactly one of the switches. On average, how many minutes will pass before all three lights first get turned on at the same time?

(A) 3 (B) 8 (C)$^{\heartsuit}$ 10 (D) 12 (E) None of the above

Solution. Denote by x_i the number of minutes that will pass since the moment when i lights are turned off. (Therefore, $x_0 = 0$ and we have to find x_3). Then

$$\begin{aligned}
x_0 &= 0 \\
x_1 &= 1 + \frac{1}{3}x_0 + \frac{2}{3}x_2 \\
x_2 &= 1 + \frac{2}{3}x_1 + \frac{1}{3}x_3 \\
x_3 &= 1 + x_2
\end{aligned}$$

For example, let us explain the second equation, for x_1. If 1 light is off now then one minute ago 0 lights were off with the probability 1/3 and 2 lights were off with the probability 2/3. Hence the equation.

This system of linear equations can easily be solved to get $x_0 = 0$, $x_1 = 7$, $x_2 = 9$ and $x_3 = 10$.

Alternative solution. The first possible time all three light switches can all be turned on is after 3 minutes. There is a probability of 2/9 that this occurs. If not, the on/off configuration of the switches at time $t = 3$ minutes is the same configuration as at $t = 1$ minutes.

Let Y be the event that all three switches are turned on at $t = 3$. Denote by $P(X|Y)$ the probability that event X occurs given that Y has occurred. Then

$$P(X) = P(X|Y)P(Y) + P(X|\text{ not } Y)P(\text{ not } Y).$$

If we denote by X_k be the event that three switches are on at minute k, we are looking for the expectation,

$$E = \sum_{k=1}^{\infty} kP(X_k)$$

Then

$$\begin{aligned} E &= \sum kP(X_k|Y)P(Y) + \sum kP(X_k|\text{ not } Y)P(\text{ not } Y) \\ &= E(X|Y)P(Y) + E(X|\text{ not } Y)P(\text{ not } Y) \\ &= 3 \cdot \frac{2}{9} + (E+2) \cdot \frac{7}{9}, \end{aligned}$$

so $E = 10$.

Problem 23. If you interchange the hour and the minute hands on the watch, how many times during the 24-hour day will you still get a legal time? (For example, this is true at noon but not true at 6 a.m.).

(A) 143 (B) 144 (C) 264 (D)$^{\heartsuit}$ 286 (E) 288

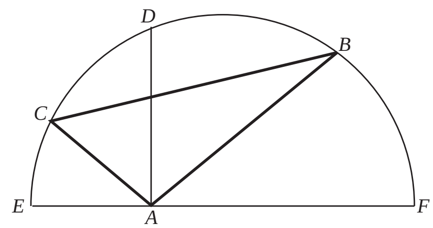
Solution. We will find the number for the period of 12 hours and then multiply the answer by two to get the answer for the 24-hour day.

Let $0 \le x < 12$ represent the hours (it need not be integral, f.e. 1 hour 25 minutes means that $x = 1\ 25/60$). Then the minute hand points at $y = 12x \mod 12$. Our condition says that the following condition must be satisfied:

$$\begin{aligned} & x = 12y \mod 12, \quad \text{i.e. } x = 144x \mod 12 \\ & 143x = 0 \mod 12 \end{aligned}$$

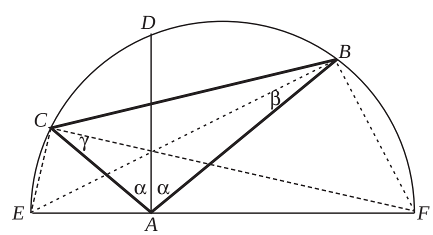
This equation has 143 solutions separated by $12/143$. For the 24-hour day, we obtain $2 \times 143 = 286$ solutions.

Problem 24. Given a semicircle with diameter $\overline{EF}$ as indicated, triangle $\triangle ABC$ with A lying on the diameter, and B and C on the semicircle. $\overline{AD}$ bisects $\angle BAC$, $\overline{BE}$ bisects $\angle ABC$, and $\overline{CF}$ bisects $\angle ACB$. If $AB = 6$, $AC = 3$, $BC = 7$, and $\overline{AD} \perp \overline{EF}$, find AD.



(A) 4 (B)$^\heartsuit$ $3\sqrt{2}$ (C) 9/2 (D) $3\sqrt{5}$ (E) None of the above

Solution. The key to this problem is one basic fact from geometry: Any triangle inscribed in a semicircle with the diameter as one side is a *right* triangle. So $\triangle EDF$ is a right triangle, and, as in the ciphering problem number 3, we have $(AD)^2 = (AE)(AF)$.



Similarly, $\triangle ECF$ and $\triangle EBF$ are right triangles. Note also that $2\alpha + 2\beta + 2\gamma = \pi$, so $\alpha + \beta + \gamma = \pi/2$.

Next, consider $\triangle AEC$ and $\triangle AFB$. $\angle CAE = \angle BAF = \pi/2 - \alpha$. $\angle ABF = \pi/2 - \beta$ and $\angle ACE = \pi/2 - \gamma$. Thus,

$$\angle ACE = \pi/2 - \gamma = \alpha + \beta = \pi - (\pi/2 - \alpha) - (\pi/2 - \beta) = \angle AFB.$$

Thus, $\triangle ACE \sim \triangle AFB$, and so $\dfrac{AE}{AC} = \dfrac{AB}{AF}$. It follows that

$$(AD)^2 = (AE)(AF) = (AB)(AC) = 18,$$

so $AD = 3\sqrt{2}$.

Problem 25. Alice, Bob, Caroline, Dave and Emily compete to see who can solve the most problems on this test. In how many different orders can they finish if ties can happen? (For example, for two people there are 3 possible outcomes, and for three, 13 outcomes.)

(A) 478 (B) 480 (C)$^{\heartsuit}$ 541 (D) 600 (E) None of the above

Solution. Call the answer for n people $f(n)$. Then $f(0) = f(1) = 1$ and

$$f(n) = \sum_{i=0}^{n-1} \binom{n}{i} f(i)$$

Indeed, there are some $0 \leq i < n$ people who did **not** score the most problems; these i people can be chosen in $\binom{n}{i}$ ways; and the number of outcomes for these i people is $f(i)$. Now, we compute:

$$\begin{aligned}
f(2) &= 1 \cdot 1 + 2 \cdot 1 = 3 \\
f(3) &= 1 \cdot 1 + 3 \cdot 1 + 3 \cdot 3 = 13 \\
f(4) &= 1 \cdot 1 + 4 \cdot 1 + 6 \cdot 3 + 4 \cdot 13 = 75 \\
f(5) &= 1 \cdot 1 + 5 \cdot 1 + 10 \cdot 3 + 10 \cdot 13 + 5 \cdot 75 = 541
\end{aligned}$$

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