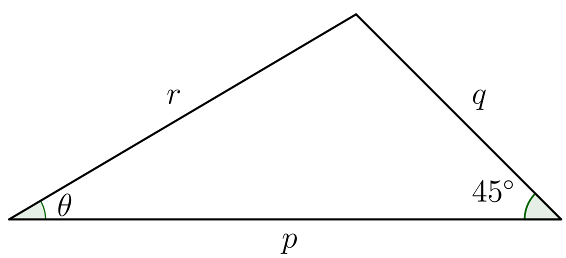
Sponsored by: UGA Math Department and UGA Math Club

Team Round / 1 hour / 210 points
October 21, 2017

WITH SOLUTIONS

Problem 1 (Double trouble)**.** Recall that the functions $c(\theta)$ and $s(\theta)$ were defined, for $0° < \theta < 135°$, as follows: Given θ, construct a triangle with angles $\theta°$ and $45°$, and side lengths as indicated in this diagram:

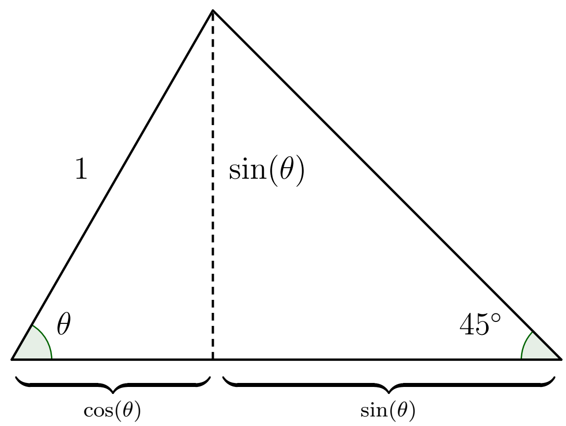


Then $c(\theta) = \dfrac{p}{r}$ and $s(\theta) = \dfrac{q}{r}$.

Find the double angle formula for $s(\theta)$, expressed as a polynomial in $c(\theta)$. For instance, a reasonable — but incorrect! — answer might look like $s(2\theta) = c(\theta)^3 + 2c(\theta)^2 - c(\theta) - 2$.

Answer. $\sqrt{2}c(\theta)^2 - \sqrt{2}$ (i.e., $\sqrt{2}(c(\theta)^2 - 1)$)

Solution. Begin by relating $s(\theta)$ and $c(\theta)$ to $\sin(\theta)$ and $\cos(\theta)$. The diagram



shows that $c(\theta) = \cos(\theta) + \sin(\theta)$, and $s(\theta) = \sqrt{2}\sin(\theta)$. Squaring the first of these shows

$$\begin{aligned} c(\theta)^2 &= \sin^2(\theta) + 2\sin(\theta)\cos(\theta) + \cos^2(\theta) \\ &= 1 + \sin(2\theta) \\ &= 1 + \frac{1}{\sqrt{2}} s(2\theta). \end{aligned}$$

So $s(2\theta) = \sqrt{2}(c(\theta)^2 - 1)$.

Problem 2 (Strike that. Reverse it.)**.** There are infinitely many pairs of distinct positive rational numbers x, y satisfying

$$x^y = y^x.$$

One example, with integers x and y, is $x = 2, y = 4$. Suppose that x, y is such a pair with neither x nor y an integer. Write $x = a/b$ and $y = c/d$, with a/b and c/d in lowest terms. (Recall that m/n is in *lowest terms* if $n > 0$ and the greatest common divisor of m and n is 1.) What is the smallest possible value of $b + d$?

Answer. 12

Solution. We can completely describe all these pairs x, y. Write $y = tx$; since we are assuming x, y are distinct, we have $t \neq 1$. The equation $x^y = y^x$

becomes $x^{tx} = (tx)^x$, and so — upon taking xth roots — $x^t = tx$. Dividing by x and raising both sides to the power $\frac{1}{t-1}$, we see that

$$x = t^{\frac{1}{t-1}}.$$

Even if t is rational, the corresponding value of x here will usually be irrational, and so will not lead to one of our sought-after pairs. One way to ensure a rational solution is to take $t = 1 + 1/k$ for some positive integer k. Then

$$x = \left(1 + \frac{1}{k}\right)^k, \quad \text{and} \quad y = tx = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k}\right)^k = \left(1 + \frac{1}{k}\right)^{k+1}.$$

If we take $k = 1$, we get the solution in integers $x = 2, y = 4$. If we take $k = 2$, the denominators are 4 and 8 (corresponding to $x = 9/4$ and $y = 27/8$). Larger values of k give even larger denominators (note that the denominator of x is k^k, while that of y is k^{k+1}); hence, the answer to the original problem is $4 + 8 = 12$, **assuming that there are no rational solutions we have missed by taking** $t = 1 + 1/k$.

Let us show that our assumption above is justified. Since we are not concerned with the ordering of our pairs, we can assume that $y > x$, and so $t > 1$. If x, y are rational, so is $t = y/x$. Write $t = p/q$, with p/q in lowest terms. Since $t > 1$, we have $p > q$. If $p = q + 1$, then $t = 1 + 1/q$, and we recover the solutions found in the last paragraph.

Suppose now that $p > q + 1$; this will lead to a contradiction, implying that the solutions we found above are all solutions. We have that $\frac{1}{t-1} = \frac{q}{p-q}$, and $\frac{q}{p-q}$ is also in lowest terms. Moreover,

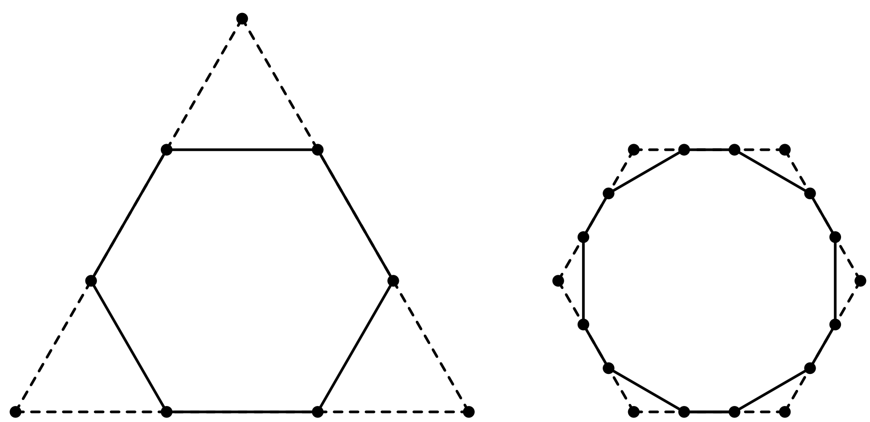
$$x = (p/q)^{\frac{q}{p-q}}.$$

For x to be rational, this last equation forces p/q to be the dth power of another positive rational number, where $d = p - q$. (This follows from unique factorization for positive integers.) So we can write $p/q = (m/n)^d$, where m/n is in lowest terms and $m > n$. Then $p/q = m^d/n^d$. Both sides are in lowest terms, forcing $p = m^d$ and $q = n^d$. But then

$$p - q = m^d - n^d = (m - n)(m^{d-1} + m^{d-2}n + \cdots + n^{d-1}).$$

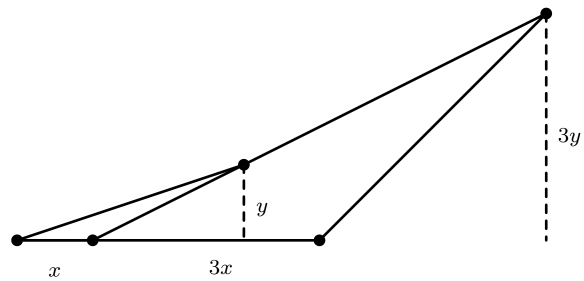
The first factor on the right is at least 1. The second is a sum of d terms, all of which are positive integers, and the first of which is at least $2^{d-1} > 1$. So the second factor is strictly larger than d. Hence, $p - q > d$. But this contradicts the definition of d.

Problem 3 (Cutting corners)**.** Begin with an equilateral triangle. Trisect each of its sides and cut off the corners. Take the resulting figure, and again trisect each of its sides and cut off the corners. If you repeat this process infinitely many times, what is the ratio of the area of the resulting figure to the area of the original triangle? The first two iterations are pictured below.



Answer. $\dfrac{4}{7}$

Solution. In each iteration, we cut off twice as many triangles as we did in the last iteration. Now we compare a $(n+1)$-th iteration triangle to a n-th iteration triangle. The $(n+1)$-th iteration triangle has a base which is $\frac{1}{3}$ the size of the n-th iteration triangle's base, and by similar triangles we find that it also has $\frac{1}{3}$ the height of that triangle. Therefore the area of an $(n+1)$-th iteration triangle is $\frac{1}{9}$ of the area of an n-th iteration triangle.



Thus, if A is the area cut off in the first iteration, then the total area which is cut off is

$$\sum_{n=0}^{\infty} A \cdot 2^n \cdot \left(\frac{1}{9}\right)^n = \frac{A}{1-\frac{2}{9}} = \frac{9}{7}A.$$

In our case, A is $\frac{1}{3}$ of the original area of the triangle, so the total area cut off is $\frac{3}{7}$ of the original triangle. This leaves the resulting figure with $\frac{4}{7}$ of the original area.

Authors. Written by Mo Hendon, Paul Pollack, and Peter Woolfitt.