University of Georgia High School Math Tournament

1785

2005

Sponsored by: UGA Math Department and UGA Math Club

Team Round / 45 min / 150 points

WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 50 points, for a total of 150 points.

For problem 3, the answer should be an exact expression, such as $\pi/2$, $\sqrt{3}+1$, $8/3$, etc. No approximate answers will be accepted.

Problem 1. (Five secret numbers) Suppose there are 5 numbers whose pairwise sums are

$$5, 9, 20, 24, 31, 35, 39, 42, 46, 61$$

What are the original 5 numbers? Write them in increasing order.

Answer.

$$-3, 8, 12, 27, 34$$

Solution. Denote the numbers $a \le b \le c \le d \le e$. Then

$a + b = 5, \qquad d + e = 61, \quad$ and

$$a+b+c+d+e = \frac{5+9+20+24+31+35+39+42+46+61}{4} = \frac{312}{4} = 78$$

Therefore,

$$c = (a + b + c + d + e) - (a + b) - (d + e) = 78 - 5 - 61 = 12$$

The next largest number after $a + b$ is $a + c = 9$, so $a = 9 - 12 = -3$. Then $b = (a + b) - a = 5 - (-3) = 8$. Similarly, $e = 34$ and $d = 27$.

Problem 2. (The last man standing) n people stand in a circle. Then, every second person is excluded until only one is left. For example, with 10 people, the order of exclusion is as follows:

$$2, 4, 6, 8, 10, 3, 7, 1, 9$$

so the last remaining person is number 5.

Now start with 2005 people. Who will be the last person standing?

Answer. 1963

Solution. Denote the answer for n by $J(n)$. Then we have:

$$J(2n) = 2J(n) - 1 \quad \text{and} \quad J(2n + 1) = 2J(n) + 1$$

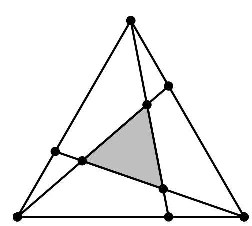
Indeed, if there are originally $2n$ people, the first n people to be eliminated are numbered $2, 4, 6, 8, \ldots, 2n$. The remaining n people, numbered $1, 3, 5, \ldots, 2n - 1$ will essentially be eliminated exactly as if they were numbered $1, 2, 3, \ldots, n$. Thus, $J(2n) = 2J(n) - 1$; the argument for the other case is similar.

To compute $J(2005)$, read down the first column and up the second:

$$\begin{aligned}
2005 &= 2\cdot 1002+1 \qquad & J(2005) &= 2\cdot 981+1 = 1963\\
1002 &= 2\cdot 501 & J(1002) &= 2\cdot 491-1 = 981\\
501 &= 2\cdot 250+1 & J(501) &= 2\cdot 245+1 = 491\\
250 &= 2\cdot 125 & J(250) &= 2\cdot 123-1 = 245\\
125 &= 2\cdot 62+1 & J(125) &= 2\cdot 61+1 = 123\\
62 &= 2\cdot 31 & J(62) &= 2\cdot 31-1 = 61\\
31 &= 2\cdot 15+1 & J(31) &= 2\cdot 15+1 = 31\\
15 &= 2\cdot 7+1 & J(15) &= 2\cdot 7+1 = 15\\
7 &= 2\cdot 3+1 & J(7) &= 2\cdot 3+1 = 7\\
3 &= 2\cdot 1+1 & J(3) &= 2\cdot 1+1 = 3\\
 & & J(1) &= 1
\end{aligned}$$

It can also be shown that if $n = (1b_{m-1}b_{m-2}\cdots b_1b_0)_2$ written in binary, then $J(n) = (b_{m-1}b_{m-2}\cdots b_1b_0 1)_2$.

Problem 3. (Two triangles) In a triangle ABC, vertices are connected to the points A', B', C' which divide the corresponding sides with the ratio 2 to 1, as in the picture, to form a small triangle KLM in the center. What is the ratio of area of ABC to the area of KLM? (The answer must be greater than 1.)



Answer. 7

Solution. Put the triangle ABC in the 3-dimensional space in the plane $x + y + z = 1$ so that the vertices are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Then the lines are cut out by planes $2x = y$, $2y = z$ and $2z = x$. Solving for $2x = y$, $2y = z$ and $x + y + z = 1$ gives $(1/7, 2/7, 4/7)$ and the other two vertices of KLM are obtained by rotating this triple around. Let the side of triangle ABC be x and the side of KLM be y. We have $x^2 = 2$. The side y is the length of the vector

$$(2/7, 4/7, 1/7) - (1/7, 2/7, 4/7) = (1/7, 2/7, -3/7)$$

Its length can be found using the distance formula:

$$y^2 = \left(\frac{1}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{-3}{7}\right)^2 = \frac{1^2 + 2^2 + 3^2}{7^2} = \frac{14}{7^2} = \frac{2}{7}$$

The ratio of the areas is

$$\frac{x^2}{y^2} = \frac{2}{2/7} = 7$$

[The ratio of these areas is independent of the shape of the original triangle: when we act on the plane by a linear transformation, all areas are multiplied by the same constant, the absolute value of its determinant.]

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