Sponsored by: UGA Math Department and UGA Math Club

Ciphering Round / 2 minutes per problem
October 22, 2016

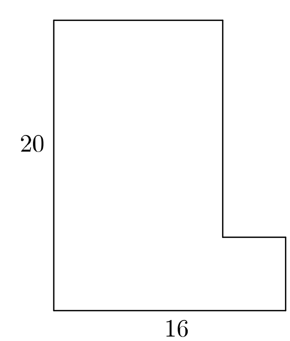
WITH SOLUTIONS

Problem 1. A rectangular prism has edge lengths $6, 8$, and 24. What is the length of the interior diagonal?

Answer. 26

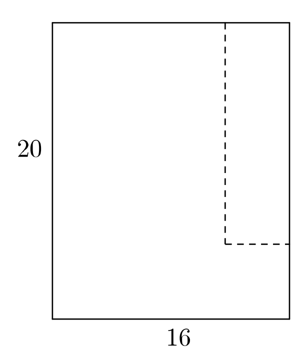
Solution. By the distance formula, the answer is $\sqrt{6^2 + 8^2 + 24^2} = \sqrt{676} = 26$. To simplify the arithmetic, use instead a box with edge lengths $3, 4$, and 12, and take advantage of the familiar Pythagorean triples $3, 4, 5$ and $5, 12, 13$.

Problem 2. What is the perimeter of the polygon? All angles are $90°$.

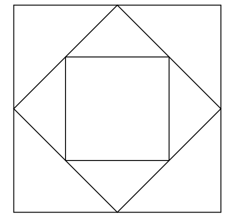


Answer. 72

Solution. The perimeter is the same as this polygon:

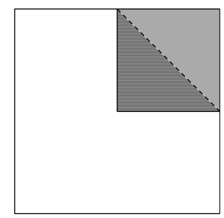


Problem 3. Start with a square. Connect the midpoints of adjacent sides to form a second square. Connect the midpoints of the second square to form a third square. What is the ratio of the area of the first square to the area of the third?

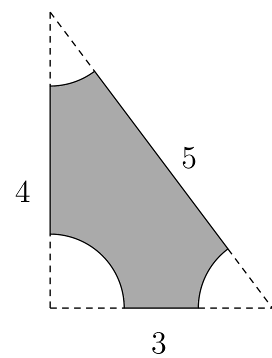


Answer. 4

Solution. In the diagram to the right, the areas of the two shaded triangles are equal. This shows that the area of the nth square is twice the area of the $(n+1)$st, and so the area of the nth square is 2^k times the $(n+k)$th. This question is the $k = 2$ case.



Problem 4. What is the area of the shaded region? The curves are arcs of circles of radius 1, centered at the vertices, and the triangle has side lengths $3, 4, 5$.



Answer. $6 - \dfrac{\pi}{2}$ or $\dfrac{12-\pi}{2}$

Solution. The area of the triangle is $\frac{1}{2} \cdot 3 \cdot 4 = 6$. The three circular sectors combine for an angle of $180°$, and so their combined area is $\frac{1}{2}\pi \cdot r^2 = \frac{\pi}{2}$. So the total area is $6 - \frac{\pi}{2}$.

Problem 5. Solve for x:

$$e^{2x} - 4e^{x+1} + 4e^2 = 0.$$

Answer. $\ln(2e)$ or $1 + \ln(2)$

Solution. This factors as $(e^x - 2e)^2$, and so $e^x = 2e$, i.e., $x = \ln(2e) = 1 + \ln(2)$.

Problem 6. 10 apples and 5 bananas cost \$45. 5 apples and 10 bananas cost \$30. What is the difference between the price of an apple and the price of a banana?

Answer. \$3 (i.e., 3 dollars)

Solution. Let A be the cost of an apple and B the cost of a banana. The given information shows that

$$10A + 5B = 45,$$
$$5A + 10B = 30.$$

Subtract, then divide by 5:

$$5A - 5B = 15,$$
$$A - B = 3.$$

Note that we did not need to solve for A and B individually.

Problem 7. If you know that the real number x is closer to 10 than to 15, and closer to 3 than to 2, what is the length of the interval in which x can lie?

Answer. 10

Solution. "Closer to 10 than to 15" simply means $x < 12.5$. Similarly, "closer to 3 than to 2" means $x > 2.5$. So x lies in the interval $(2.5, 12.5)$, which has length 10.

Problem 8. Mo gave a 10 question quiz to 40 students. So far he has graded all of the questions on $\frac{1}{5}$ of the quizzes, and $\frac{1}{5}$ of the questions on the rest of the quizzes. What percentage of the questions has he graded?

Answer. 36%

Solution. You can simply compute the number of questions (out of 400 total) that have been graded:

$$\text{all of the questions on } \frac{1}{5} \text{ of the quizzes: } 10 \cdot \frac{1}{5} \cdot 40 = 80.$$

$$\frac{1}{5} \text{ of the questions on the rest of the quizzes: } (\frac{1}{5} \cdot 10) \cdot (\frac{4}{5} \cdot 40) = 64.$$

So he's graded $\frac{144}{400} = 36\%$ of the questions.

Alternatively, you can think in terms of probability. A problem has not been graded if it's not on one of the quizzes that's been completely graded ($p = 4/5$) *and* it's not in the 1/5 of the problems on each test that have been graded (also $p = 4/5$). So it's not been graded with probability $(4/5)^2 = 16/25 = 64\%$; i.e., 36% of problems have been graded.

Problem 9. If x, y, and z are positive integers satisfying $2^x + 2^y = 2^z$ and $2^x \cdot 2^y = 2^{z+1}$, then what is z?

Answer. 3 (or $z = 3$)

Solution. We observe first that $2^x + 2^y = 2^z$ implies that $x = y$. For example, if $x > y$, then $2^x + 2^y = 2^y(2^{x-y} + 1)$, which would make $2^{x-y} + 1$ an odd factor of 2^z. Similarly, we cannot have $x < y$.

Since $x = y$, we have $2^z = 2^x + 2^x = 2^{x+1}$, so that $z = x+1$. Substituting $x = y = z - 1$ into the second given equation yields $2^{2z-2} = 2^{z+1}$, and hence $2z - 2 = z + 1$, so that $z = 3$.

Problem 10. What is the smallest positive integer that has exactly 10 factors? (Hint: It is not 6, which has 4 factors: 1, 2, 3, and 6.)

Answer. 48

Solution. A reasonable first guess is 2^9, which has the 10 divisors $2^0, 2^1, \ldots, 2^9$. But $2^4 \cdot 3$ is smaller and has the 10 divisors $2^0, 2^1, 2^2, \ldots, 2^4, 2^0 \cdot 3, 2^1 \cdot 3, \ldots, 2^4 \cdot 3$. To see that $2^4 \cdot 3$ is the smallest possible, recall that if $p_1, \ldots, p_k$ are distinct primes and $e_1, \ldots, e_k$ are positive integers, then the number of divisors of n is $(e_1+1) \cdots (e_k+1)$. So a number with 10 divisors is either of the form p^9 for a prime p, or of the form $p^4 q$ for distinct primes p, q. The smallest solution of the first kind is clearly 2^9, while the smallest solution of the second kind is $2^4 \cdot 3^1 = 48$.

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