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Ciphering Round / 2 minutes per problem
October 24, 2015

WITH SOLUTIONS

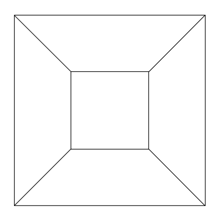
Problem 1. If $a+b$ means the maximum of a and b, and $a \cdot b$ means their sum, what is $(2+3)\cdot(4+5)$?

Answer. 8

Solution. $(2+3)\cdot(4+5) = \max\{2,3\} + \max\{4,5\} = 3+5 = 8$.

Remark: *Tropical geometry* is an area of mathematics which studies the geometric and combinatorial properties of tropical polynomials. A *tropical polynomial* is nothing more than a usual polynomial where addition and multiplication are as described above. Can you draw the graph of the tropical polynomial $P(x) = 2\cdot x+1$? Tropical geometry turns out to have important interactions with biology.

Problem 2. An x by x square is drawn in the center of a 1 by 1 square, and then the corners are connected as shown. If the 5 regions all have the same area, what is x?



Answer. $\frac{1}{\sqrt{5}}$ or $\frac{\sqrt{5}}{5}$

Solution. If each region has the same area, then each has area $1/5$, so $x^2 = 1/5$, i.e., $x = 1/\sqrt{5}$.

Problem 3. If the average of a and b is 20, the average of b and c is 30, and the average of a and c is 70, what is the average of a, b, and c?

Answer. 40

Solution. The given averages imply that $a+b=40$, $b+c=60$, and $a+c=140$. Adding these together, we get $2a+2b+2c=240$, so $\frac{a+b+c}{3}=\frac{240}{6}=40$.

Problem 4. In this problem, $\log(x)$ denotes the base 10 logarithm of x. Simplify the sum

$$\sum_{k=1}^{9} \log\left(1+\frac{1}{k}\right) = \log\left(1+\frac{1}{1}\right) + \log\left(1+\frac{1}{2}\right) + \log\left(1+\frac{1}{3}\right) + \cdots + \log\left(1+\frac{1}{9}\right).$$

Answer. 1

Solution.

$$\begin{aligned}&\log\left(1+\frac{1}{1}\right) + \log\left(1+\frac{1}{2}\right) + \log\left(1+\frac{1}{3}\right) + \cdots + \log\left(1+\frac{1}{9}\right)\\ &\quad = \log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \cdots + \log\left(\frac{10}{9}\right) = \log\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{10}{9}\right),\end{aligned}$$

which is $\log(10)=1$.

Remark: *Benford's law* says that in naturally occurring numerical data sets, the leading digit 1 should appear with frequency $\log(1+1/1)\approx 30.1\%$, the leading digit 2 with frequency $\log(1+1/2)\approx 17.6\%$, and similarly for the digits 3 through 9. Note that this explains why the sum above must come out to 1 ! Despite its name, Benford's law was originally discovered by astronomer Simon Newcomb, who wrote:

> That the ten digits do not occur with equal frequency must be evident to any one making much use of logarithmic tables, and noticing how much faster the first pages wear out than the last ones.

The law was rediscovered by physicist Frank Benford in 1938. The IRS is rumored to use Benford's law in detecting tax fraud.

Problem 5. On a recent backpacking trip, slow Mo hiked 9 miles per day for the first 6 days. Then fast Dave joined, and they each hiked 16 miles per day for the next 4 days. How many miles per day did Mo average for the entire trip?

Answer. 11.8 (miles/day)

Solution. Total miles $= 9\,\frac{\text{mi}}{\text{day}} \times 6\,\text{days} + 16\,\frac{\text{mi}}{\text{day}} \times 4\,\text{days} = 118\,\text{miles}$.
Total time = 6 days + 4 days = 10 days.
So Mo's average was $\frac{118\,\text{mi}}{10\,\text{days}} = 11.8$ miles/day.

Problem 6. A cube with side length 1 is inscribed in a sphere. What is the radius of the sphere?

Answer. $\frac{\sqrt{3}}{2}$

Solution. The diagonal of the cube is a diameter of the sphere. The diagonal has length $\sqrt{1^2+1^2+1^2} = \sqrt{3}$, and so the radius is $\frac{\sqrt{3}}{2}$.

Problem 7. How many ways can 7 people be split into two groups, if each group must contain at least 2 people?

Answer. 56 (ways)

Solution. The smaller of the two groups will have either 2 or 3 people. These groups can be chosen in $\binom{7}{2}$ or $\binom{7}{3}$ ways, respectively. So the total number of ways is

$$\binom{7}{2} + \binom{7}{3} = \frac{7\cdot 6}{2\cdot 1} + \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 21 + 35 = 56.$$

Remark: Dividing a group of seven people into two groups with at least two people each counts some particular subspaces of the space $\overline{M}_{0,7}$ called "boundary divisors". $\overline{M}_{0,n}$ (where now $n \geq 3$ is any integer) is a space whose geometry is connected to the geometry of trees of lines with n marked points. $\overline{M}_{0,n}$ has very intricate combinatorial properties mathematicians are still trying to discover.

Problem 8. Let $\overleftarrow{n}$ denote the digit reversal of the natural number n, so that, for example, $\overleftarrow{123} = 321$. Find

$$(10 + 11 + \cdots + 99) - (\overleftarrow{10} + \overleftarrow{11} + \cdots + \overleftarrow{99}).$$

Answer. 405

Solution. If a and b are both nonzero, then ab will cancel $\overleftarrow{ba}$, so we only need to consider

$$\begin{aligned}(10+20+\cdots+90)-(\overleftarrow{10}+\overleftarrow{20}+\cdots+\overleftarrow{90}) &= (10+20+\cdots+90)-(1+2+\cdots+9)\\ &= 10(1+2+\cdots+9)-(1+2+\cdots+9)\\ &= 450-45\\ &= 405.\end{aligned}$$

Problem 9. What is the coefficient of x^{25} in

$$\prod_{k=0}^{\infty}(1+x^{2^k}) = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\cdots \quad ?$$

Answer. 1

Solution. An x^{25} must come from a term $x^{2^{k_1}}x^{2^{k_2}}\cdots$, with $2^{k_1}+2^{k_2}+\cdots = 25$. Since 25 can be written as a sum of distinct powers of 2 in only one way (by the uniqueness of binary expansions), there is only one x^{25} in the product.

Problem 10. What is the base 10 representation of the binary number

$$11111011111 \quad ?$$

(That's 5 ones, a zero, and 5 more ones.)

Answer. 2015

Solution.

$$\begin{aligned}11111011111 &= 11111111111 - 100000\\ &= (2^{11}-1)-32\\ &= 2047-32\\ &= 2015.\end{aligned}$$

Authors. These problems were written by Mo Hendon, Paul Pollack, and Luca Schaffler.