1785

Sponsored by: UGA Math Department and UGA Math Club

Ciphering Round / 2 minutes per problem
October 17, 2009

WITH SOLUTIONS

No calculators are allowed on this test. 2 minutes per problem, 10 points for each correct answer.

Problem 1. What is the smallest positive integer n so that each of n and $n+2$ is a product of two distinct primes?

Answer. 33

Solution. We need n to be odd, so we try $n = 3k$ for various *prime* values of k. With $k = 5$, $n = 15$ and 17 is prime. With $k = 7$, $n = 21$ and 23 is prime. With $k = 11$, $n = 33$ and $35 = 5 \cdot 7$. Success. (If the smaller number were *not* divisible by 3, it would have to be at least $5 \cdot 7 = 35$, and this is already too large, as we've seen.)

Problem 2. Four red points and four blue points are located in the plane. How many ways can you draw four line segments, each starting at a red point and ending at a blue point, if you must use each point exactly once?

Answer. 24

Solution. We have 4 blue points to which we can join the first red point, 3 to which we can join the second, 2 the third, and 1 the last. This gives $4! = 24$ ways to complete the task.

Interestingly, if no three of the original points are collinear, there is a way to achieve the task with no crossings. Is the same true with more dots?

Problem 3. Eight points are located in the plane. How many ways can you draw four line segments, each starting at one of the points and ending at another, if you must use each point exactly once?

Answer. 105

Solution. The first point can be joined to 7 possible points; the third point can be joined to 5 possible points, etc., obtaining $7 \cdot 5 \cdot 3 \cdot 1 = 105$ possible solutions.

Alternatively, there are $\binom{8}{2} \cdot \binom{6}{2} \cdot \binom{4}{2}$ total pairs, but we must divide by $4!$, since we do not care about what order we choose the pairs.

Problem 4. The faces of a rectangular box have areas 10, 14, and 35 sq ft, respectively. What is the volume of the box, in cu ft?

Answer. 70

Solution. If x, y, and z are the dimensions of the box, we have $xy = 10$, $xz = 14$, and $yz = 35$. Therefore, $xyz = \sqrt{(xy)(xz)(yz)} = \sqrt{(10)(14)(35)} = \sqrt{(2 \cdot 5)(2 \cdot 7)(5 \cdot 7)} = 2 \cdot 5 \cdot 7 = 70$.

Problem 5. The dresser drawer of a faithful UGA student contains 5 red socks and 3 black socks. He pulls out two socks at random (without replacement). What is the probability that the socks have the same color? Give your answer as a fraction in lowest terms.

Answer. $13/28 \approx 46\%$

Solution. There are $8 \cdot 7$ ways to draw two socks. There are $5 \cdot 4$ ways to draw two red socks and $3 \cdot 2$ ways to draw two black socks. This gives a probability of $26/56 = 13/28$ of drawing matching socks.

Alternatively, we have $\binom{8}{2} = 28$ pairs, of which $\binom{5}{2} = 10$ are both red and $\binom{3}{2} = 3$ are both red. This gives $13/28$.

Problem 6. Find the radius of the circle passing through the three points $(2, 3)$, $(2, 6)$, and $(6, 3)$.

Answer. $5/2$

Solution. This is a triangle with sides 3, 4, and 5, so it is a right triangle, and the center of the circumscribed circle is at the midpoint of the hypotenuse.

Problem 7. What is the smallest positive integer that is the hypotenuse of two distinct Pythagorean triangles? (That is, we want the smallest positive integer z so that $z^2 = x^2 + y^2$ for two different *pairs* of positive integers x and y.)

Answer. 25

Solution. It is well known that 7, 24, 25 is a Pythagorean triple. Multiplying the usual 3, 4, 5 by 5, we get 15, 20, 25.

Every "reduced" Pythagorean triple can be expressed in the form $x = m^2 - n^2$, $y = 2mn$, $z = m^2 + n^2$ for some positive integers m and n with $m > n$. We obtain 25 by choosing $m = 4$ and $n = 3$. To get a smaller z, we will need $m \leq 4$, $n < 3$, and the list of z's we so obtain is $z = 5, 10, 13, 17, 20$, but there's no way to get these two ways.

The next such z is 50, but far more interesting is $z = 65$, and in this case there are *four* different pairs:

$$65^2 = 16^2 + 63^2 = 33^2 + 56^2 = 25^2 + 60^2 = 39^2 + 52^2 \quad !!$$

(The last two are less interesting, of course, but they're there.) Note that $65 = 5 \cdot 13$, each of which is a Pythagorean z. The mystery is best resolved by using complex numbers (officially, Gaussian integers):

$$\begin{aligned} 65^2 &= \left(3^2+4^2\right)\left(5^2+12^2\right) = \big((3+4i)(3-4i)\big)\big((12+5i)(12-5i)\big) \\ &= \big((3+4i)(12+5i)\big)\big((3-4i)(12-5i)\big) = (16+63i)(16-63i) = 16^2+63^2 \end{aligned}$$

but also

$$= \big((3-4i)(12+5i)\big)\big((3+4i)(12-5i)\big) = (56-33i)(56+33i) = 56^2+33^2\,.$$

Problem 8. Find the 2009^{th} digit of the decimal expansion of $4/7$.

Answer. 2

Solution. The decimal expansion of $k/7$ has period 6. We note that $2009 \pmod 6 \equiv 5$ (either divide to get 334 with a remainder of 5, or, by inspection, see that $2009 \pmod 2 \equiv 1$ and $2009 \pmod 3 \equiv 2$), so we look at

$$\frac{4}{7} = 0.\overline{571428}$$

and see that the fifth digit is 2.

Problem 9. If a and b are the roots of $x^2+px+q=0$, then express $a^3-a^2b-ab^2+b^3$ in terms of p and q alone.

Answer. $-p^3+4pq$

Solution. We have $x^2+px+q=(x-a)(x-b)$, so $p=-(a+b)$ and $q=ab$. Therefore

$$a^3-a^2b-ab^2+b^3 = (a+b)^3-4(a^2b+ab^2) = (a+b)^3-4ab(a+b) = -p^3+4pq\,.$$

Problem 10. A sphere of radius 1 is inscribed in a right circular cone with base radius 2. What is the height of the cone?

Answer. 8/3

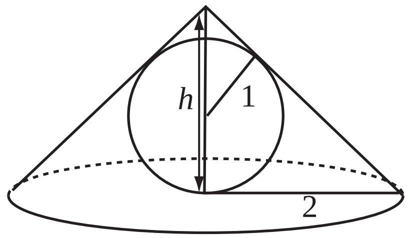
Solution. As pictured, we have similar right triangles with

$$\frac{1}{h-1} = \frac{2}{\sqrt{h^2+4}}\,,$$

and so

$$2(h-1) = \sqrt{h^2+4} \implies 4(h^2-2h+1) = h^2+4 \implies 4h(h-2) = h^2\,.$$

Since $h \neq 0$, we have $h = 8/3$.



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