Sponsored by: UGA Math Department and UGA Math Club

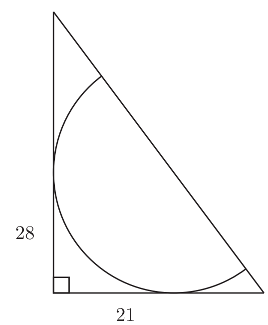
Written test, 25 problems / 90 minutes

October 24, 2015

Problem 1. How many prime numbers can be written both as a sum and as a difference of two prime numbers?

(A) 0 (B) 1 (C) 2 (D) 4 (E) infinitely many

Problem 2. A semicircle is drawn in a right triangle, tangent to both legs, and with a diameter on the hypotenuse. If the legs have length 21 and 28, what is the radius of the semicircle?

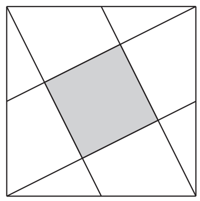


(A) 10 (B) 11 (C) 12 (D) 13 (E) 14

Problem 3. Priya drove her Prius for 30 minutes at 60 miles/hour (mph), and her gas mileage was 45 miles/gallon (mpg). She then slowed down to 50 mph for the next hour, and her fuel consumption for that hour improved to 50 mpg. What was Priya's average fuel consumption, in mpg, for the entire trip?

(A) 46 (B) $47\frac{3}{11}$ (C) $47\frac{1}{2}$ (D) 48 (E) $48\frac{1}{3}$

Problem 4. In the diagram shown, each vertex of the larger square is connected to the midpoint of a side.



If the larger square has area 1, what is the area of the shaded square?

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{5}$ (D) $\frac{1}{6}$ (E) none of these

Problem 5. For how many natural numbers $n \leq 25$ is $(n-1)!$ not divisible by n?

(A) 9 (B) 10 (C) 12 (D) 13 (E) 14

Problem 6. If r_1 and r_2 are the (real or complex) solutions to $ax^2+bx+c=0$, what is $r_1^2+r_2^2$?

(A) $\frac{b^2-2ac}{a^2}$ (B) $\frac{b^2+2ac}{a^2}$ (C) $\frac{-b^2+2ac}{a^2}$ (D) $\frac{b^2-4ac}{a^2}$ (E) $\frac{b^2-4ac}{2a}$

Problem 7. You have to climb a staircase with infinitely many steps according to the following pattern: 11 steps up, 8 steps down, repeat. If you start at the first step, how many times will you pass the 2015th step of the staircase?

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

Problem 8. A confused student wanted to solve an equation of the form

$$(x-3)(b-x)=6$$

for the variable x. He tried to find the two solutions by instead solving the two equations

$$x-3=6$$
$$b-x=6.$$

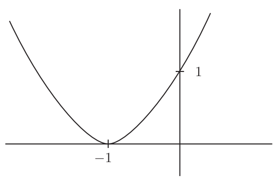
Surprisingly, he got both solutions correct! What is b?

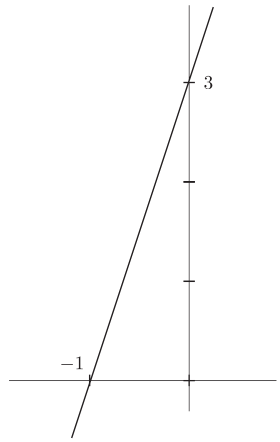
(A) 0 (B) 3 (C) 6 (D) 9 (E) 10

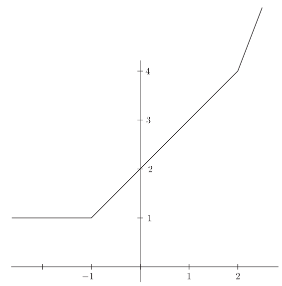
Problem 9. How many polynomials with nonnegative integer coefficients have $p(10)=200$?

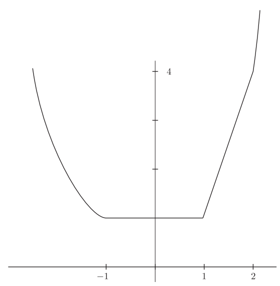
(A) 22 (B) 31 (C) 32 (D) 33 (E) infinitely many

Problem 10. In *tropical arithmetic*, $a+b$ means the maximum of a and b, while $a \cdot b$ means the sum of a and b. So, for example, $2+3=3$ and $5^2=10$. What is the graph of the tropical polynomial x^2+2x+1?

(A) 

(B) 

(C) 

(D) 

(E) none of the above

Problem 11. Suppose `ab` is a 2 digit number with the property that the 6 digit number `1234ab` is divisible by 9 and `ab1234` is divisible by 11. What is a^2-b^2?

(A) 16 (B) 34 (C) -11 (D) -72 (E) -153

Problem 12. In writing this test, we create 25 problems and then rank them. 10 should be "easy", 10 "medium", and 5 "hard". When doing this, we wondered how many ways there are to put the 25 problems into these 3 groups.

Luca claimed that it could be done in $\binom{25}{15}\binom{15}{10}$ ways.

Mo claimed that it could be done in $\binom{25}{10}\binom{15}{10}$ ways.

Paul claimed that it could be done in $\binom{25}{5}\binom{20}{10}$ ways.

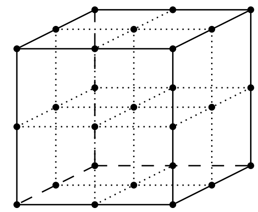
Ted claimed that it could be done in $\binom{25}{10}\binom{15}{5}$ ways.

Here $\binom{n}{k}$ is the binomial coefficient, representing the number of ways of choosing k elements from an n-element set.

How many of these claims are correct?

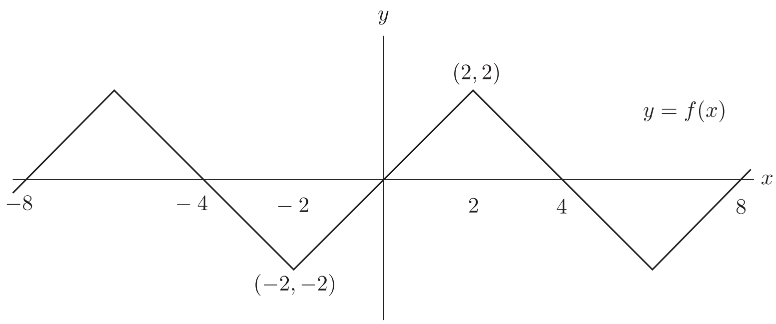
(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Problem 13. Suppose you want to cover the cube in the figure with tetrahedra having the black dots as possible vertices. What is the maximum number of tetrahedra you can use assuming that the intersection of two tetrahedra is either empty, a vertex, an edge, or a face of both of them?

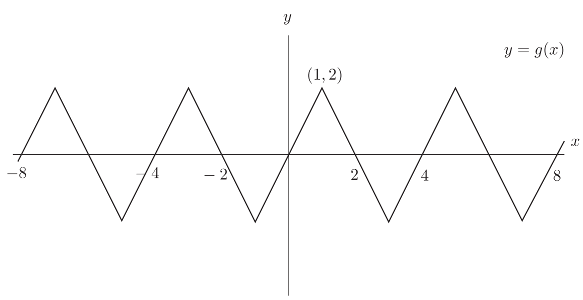


(A) 8 (B) 24 (C) 32 (D) 48 (E) 64

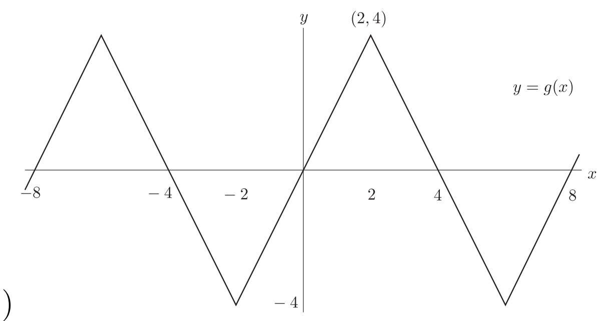
Problem 14. The following is the graph of a certain function $f(x)$:



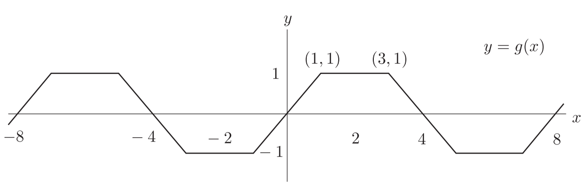
Which of the following is the graph of $g(x) = \dfrac{f(x+1)+f(x-1)}{2}$?



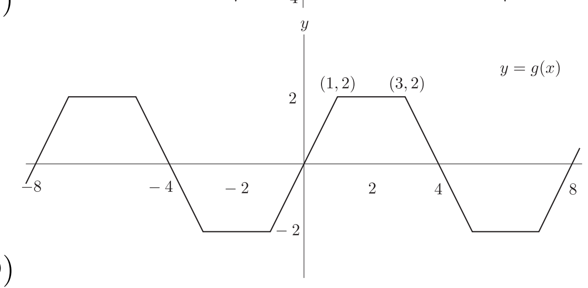
(A)



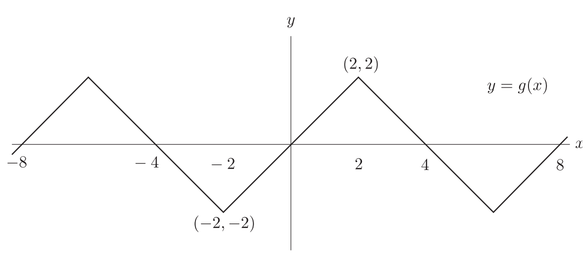
(B)



(C)



(D)



(E)

Problem 15. A regular tetrahedron of edge length 1 is inscribed in a sphere. What is the radius of the sphere?

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{\sqrt{6}}{4}$ (C) $\frac{\sqrt{6}}{2}$ (D) 1 (E) $\sqrt{3}$

Problem 16. Which of the following tests will determine whether a number, written in base 3, is even (divisible by 2)?

I. the rightmost digit is 0 or 2,

II. the sum of the digits is even,

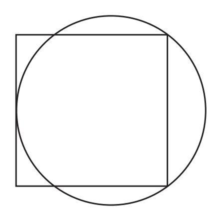
III. the alternating sum of the digits is even.

(A) I only (B) II only (C) III only (D) I and II only
(E) II and III only

Problem 17. At almost every point P on the circle, there is a unique line $y = mx + b$ which is tangent to the circle at P. Of course, m and b depend on P. If we plot the points $(m(P), b(P))$ in the (m, b)-plane, what is the shape of the resulting curve?

(A) circle (B) union of lines (C) ellipse but not a circle (D) parabola (E) hyperbola

Problem 18. Start with a circle of radius 1, and draw a square so that two adjacent corners of the square lie on the circle, and the opposite side of the square is tangent to the circle. What is the side length of the square?



(A) 2 (B) $\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $\frac{8}{5}$ (E) $\frac{13}{8}$

Problem 19. Begin with a solid cylinder of radius r and height h. Intersect with a plane that contains the diameter of the top circle and is tangent to the bottom circle. What is the area of the intersection?

(A) πrh (B) $\pi r\sqrt{r^2 + h^2}$ (C) $r\sqrt{r^2 + h^2}$ (D) $\frac{1}{2}\pi rh$ (E) $\frac{1}{2}\pi r\sqrt{r^2 + h^2}$

Problem 20. Suppose you rotate the graph of $y = x^3$ clockwise through an angle of $45°$ around the origin. The resulting curve is the graph of what polynomial?

(A) $x^3 - x$ (B) $x^3 - 2x$ (C) $\frac{1}{2}x^3 - x$ (D) it is not the graph of a function
(E) it is the graph of a function but not a polynomial

Problem 21. Evaluate

$$\sum_{1\le n\le 100} \left\lfloor \sqrt{100/n} \right\rfloor = \left\lfloor \sqrt{100/1} \right\rfloor + \left\lfloor \sqrt{100/2} \right\rfloor + \cdots + \left\lfloor \sqrt{100/100} \right\rfloor .$$

Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(A) 153 (B) 167 (C) 180 (D) 199 (E) 200

Problem 22. If you list in increasing order all the positive integers that can be written as the sum of distinct nonnegative integer powers of 3, what number is 50th on the the list?

(A) 283 (B) 327 (C) 337 (D) 356 (E) 364

Problem 23. Evaluate

$$\sum_{j=0}^{\infty} \frac{(1/3)^{2^j}}{1-(1/3)^{2^{j+1}}} = \frac{1/3}{1-(1/3)^2} + \frac{(1/3)^2}{1-(1/3)^4} + \frac{(1/3)^4}{1-(1/3)^8} + \cdots .$$

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) 1 (D) $\frac{3}{2}$ (E) $\frac{4}{3}$

Problem 24. How many odd numbers are there in the 125th row of Pascal's triangle? (Here we number so that the first row is the row $1, 1$.)

(A) 2 (B) 32 (C) 63 (D) 64 (E) 126

Problem 25. A group of math students decided to play a game of `BizzBuzz`. Here are the rules:

Players sit in a circle and take turns saying either a number or a word.
The first player must start with 1.
The nth player must say the number n, **except**:

- if n is even, she must say Bizz,
- if n is divisible by 3, she must say Buzz,
- if n is divisible by both 2 and 3, she must say BizzBuzz,
- if n is divisible by 5, she says n.

Each rule overrules the preceding. The game starts like this: 1, Bizz, Buzz, Bizz, 5, BizzBuzz, 7, Bizz, Buzz, 10, 11, BizzBuzz, 13, Bizz, 15. What is the 2015th number that will be said (assuming correct play)?

(A) 4030 (B) 4319 (C) 4320 (D) 4321 (E) 6045