Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

November 8, 2014

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

   where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit quietly.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

Problem 1. How many ordered triples of prime numbers (x, y, z) are there with $x^y - z = 1$ and and $z \leq 2014$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 2. Two points A and B lie on the graph of $y = x^3$. If their x-coordinates differ by 1, what is the least that their y-coordinates can differ?

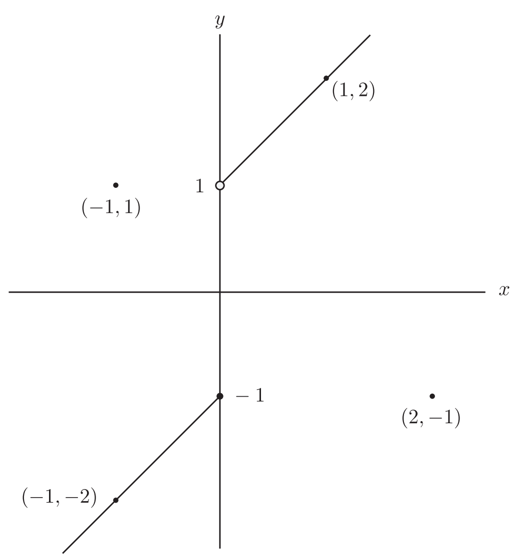
(A) 0 (B) $\frac{1}{8}$ (C) $\frac{1}{4}$ (D) $\frac{1}{2}$ (E) 1

Problem 3. Which real number is the value of the following infinite product?

$$(1 + \frac{1}{2}) \times (1 + \frac{1}{2^2}) \times (1 + \frac{1}{2^4}) \times (1 + \frac{1}{2^8}) \times \cdots$$

(A) 1 (B) $\frac{1+\sqrt{5}}{2}$ (C) 2 (D) e (E) diverges to infinity

Problem 4. In "shift geometry", a line shifts vertically 2 units as it crosses the y-axis, then continues with the same slope. For example, the "line" from $(-1, -2)$ to $(1, 2)$ is as shown.



Where does the "shifted" line from $(-1, 1)$ to $(2, -1)$ intersect the line shown?

(A) they do not intersect (B) $(\frac{4}{5}, -\frac{1}{5})$
(C) $(-\frac{2}{5}, \frac{3}{5})$ (D) $(\frac{2}{7}, \frac{9}{7})$, (E) $(\frac{1}{8}, \frac{1}{4})$

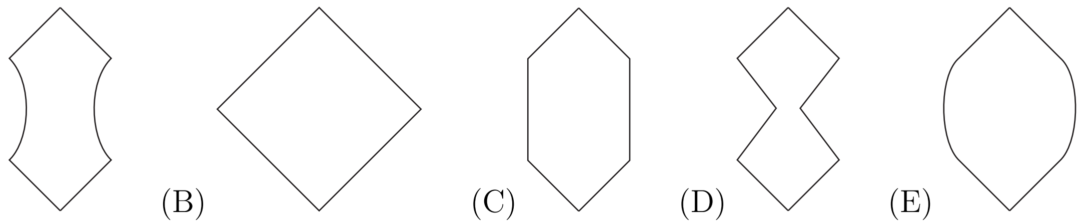
Problem 5. If a_n is defined recursively by $a_1 = 1$ and $a_{n+1} = \frac{1}{3}a_n$ for $n \geq 1$, then $a_n \to 0$ as $n \to \infty$. Find a value of c so that, if b_n is defined recursively by

$$b_n = 1, \quad b_{n+1} = \frac{1}{3}b_n + c \quad \text{for } n \geq 1,$$

then $b_n \to 2014$ as $n \to \infty$.

(A) $\frac{4028}{3}$ (B) $\frac{2014}{3}$ (C) $\frac{1007}{3}$ (D) 2014 (E) there is no such c

Problem 6. Suppose you rotate a cube rapidly around one of its diagonals. Which of the following most closely resembles the silhouette of the resulting solid of revolution?



(A) (B) (C) (D) (E)

Problem 7. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n. We call f **eponymous** if $f(0) = a_0, f(1) = a_1, \ldots, f(n) = a_n$. (Of course, $f(0) = a_0$ is true for all polynomials; the other conditions are typically not true.) Now suppose $f(x)$ is an eponymous polynomial of degree 2 and $f(0) = 1$. Find $f(3)$.

(A) -5 (B) -3 (C) -1 (D) 1 (E) 3

Problem 8. Erect a pole of length 1 on a sphere of radius 2, not necessarily perpendicular to the surface. Now shine a light so that the shadow of the pole on the sphere is as long as possible. How long is the shadow?

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{2\pi}{3}$ (E) π

Problem 9. Tetra and Yuri both want to approximate $\sum_{n=1}^{2014} \frac{1}{n}$. Tetra approximates each $\frac{1}{n}$ by rounding it up to the nearest $\frac{1}{10}$; for example, she approximates $\frac{1}{3}$ as 0.4. Yuri approximates each $\frac{1}{n}$ by rounding down to the nearest $\frac{1}{10}$. What is the difference of their sums?

(A) 201 (B) 201.4 (C) 2010 (D) 2014 (E) $\ln(2014)$

Problem 10. Suppose x is chosen randomly from the interval $(0, 1)$. What is the probability that the leftmost digit of $\frac{1}{x}$ is 1?

(A) $\frac{2}{3}$ (B) $\frac{5}{9}$ (C) $\frac{1}{2}$ (D) $\frac{1}{9}$ (E) $\frac{1}{10}$

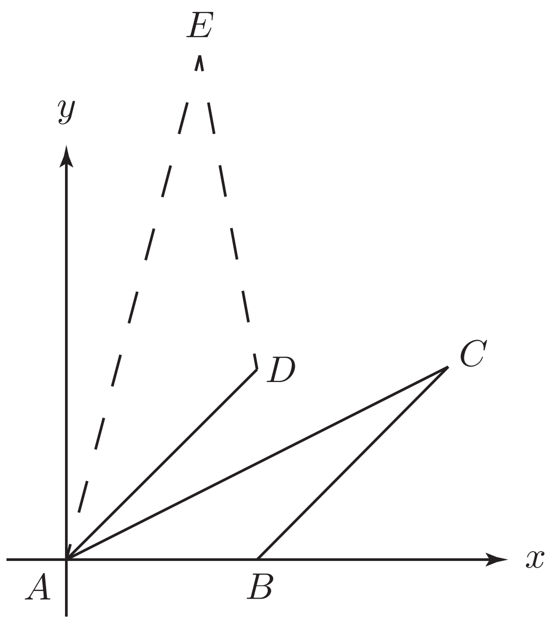
Problem 11. What is the largest integer n for which $\dfrac{20!}{1!2!3!4!n!}$ is an integer?

(A) 5 (B) 10 (C) 11 (D) 13 (E) 15

Problem 12. A **Pythagorean triangle** is one with integer side lengths a, b, c satisfying $a^2 + b^2 = c^2$. What is the smallest positive integer that does *not* occur as the radius of a circle inscribed in a Pythagorean triangle?

(A) 2 (B) 3 (C) 4 (D) 5 (E) every positive integer occurs

Problem 13. Suppose $A = (0, 0)$, $B = (2, 0)$, $C = (4, 2)$, and $D = (2, 2)$. For which of the following points E is ΔABC similar to ΔADE?



(A) $(2, 4)$, (B) $(2, 5)$, (C) $(2, 6)$, (D) $(1, 4)$, (E) $(3, 5)$

Problem 14. A square formation of Army cadets, 50 feet on the side, is marching forward at a constant pace. The company mascot, a bulldog, starts at the center of the rear rank, trots forward in a straight line to the center of the front rank, then trots back again in a straight line to the center of the rear. At the instant he returns to his original position, the cadets have advanced exactly 50 feet. Assuming the dog trots at a constant speed and loses no time in turning, how many feet does he travel?

(A) 50 ft (B) $50 + 50\sqrt{2}$ ft (C) 100 ft (D) $100 + 50\sqrt{2}$ ft (E) 150 ft

Problem 15. Fran the Frog is resting at Lilypad #0 in the middle of an infinite, bidirectional sequence of Lilypads numbered with the integers. Fran has the ability to jump forwards or backwards, but can only move by a square number of steps at each jump. For example, in 3 moves, Fran could jump 100 lilypads forward to 100, then 144 lilypads backward to -44, then 64 lilypads forward to 20. But this is not the shortest way of reaching 20, since Fran could have jumped forward 16 lilypads and then another 4 lilypads to get to 20 in just 2 moves.

If Fran wants to reach 2014 instead of 20, what is the smallest number of moves Fran can make?

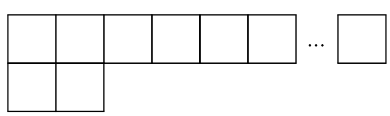
(A) 2 (B) 3 (C) 4 (D) 5 (E) more than 5

Problem 16. In $\mathbb{R}^n$, draw spheres of radius 1 at each of the points $(\pm 1, \pm 1, \ldots, \pm 1)$. Notice that each of these 2^n spheres is tangent to the adjacent spheres, and also tangent to (but contained in) the cube with vertices $(\pm 2, \pm 2, \ldots, \pm 2)$. Now draw one more sphere, centered at the origin, with the largest radius subject to the condition that no point of the other 2^n spheres is inside the central sphere. What is the smallest value of n for which the central sphere extends outside the cube?

(A) 5 (B) 8 (C) 9 (D) 10 (E) the central sphere never extends outside the cube

Problem 17. What is the smallest value of n so that $\sum_{k=1}^{n} \arctan(k) \geq 2\pi$.

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 18. 

Suppose a total of n squares are arranged as shown, $n \geq 4$. In how many ways can the numbers $1, 2, \ldots, n$ can be placed in the box so that both rows are increasing left to right, and all columns are increasing top to bottom?

(A) $\frac{n(n-3)}{2}$ (B) $\frac{n^2-5n+8}{2}$ (C) $\frac{n(n-1)}{2}$ (D) $\frac{n^2-3n+2}{2}$ (E) none of the above

Problem 19. Evaluate the product $\prod_{k=1}^{45}(1 + \tan(k^\circ))$.

(A) $(\frac{1+\sqrt{5}}{2})^{23}$ (B) 3^{15} (C) $2^{45/2}$ (D) π^{14} (E) none of the above

Problem 20. Consider a collection of airports at distinct distances from each other. A plane leaves each airport and flies to the nearest other airport. What is the most planes that could land at the same airport?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 21. For how many nonnegative integers $m < 2014$ is the polynomial

$$1 + x^{2014} + x^{2 \cdot 2014} + \cdots + x^{2014 \cdot m}$$

evenly divisible by the polynomial

$$1 + x + \cdots + x^m?$$

(A) 1 (B) 936 (C) 1007 (D) 2013 (E) 2014

Problem 22. Evaluate

$$\sum_{m=1}^{100} \left\lfloor \cos^2 \left(\pi \cdot \frac{(m-1)! + 1}{m} \right) \right\rfloor.$$

Here $\lfloor x \rfloor$ is the *floor function* of x, i.e., the largest integer less than or equal to x.

(A) 1 (B) 12 (C) 25 (D) 26 (E) 100

Problem 23. Let $f(x) = 2x^2 - 1$, and let $f^{(k)}(x)$ denote the kth iterate of $f(x)$. For how many distinct real values of t is $f^{(2014)}(t) = 1$?

(A) 1 (B) 3 (C) $2^{2013} + 1$ (D) $2^{2014} - 1$ (E) 2^{2014}

Problem 24. Consider a triangle in the plane whose vertices have integer coordinates. Recall that Pick's Theorem says that the area of this triangle is

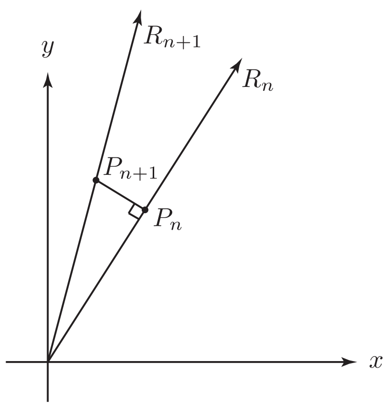
$$A = I + \frac{B}{2} - 1$$

where I is the number of integer points in the interior of the triangle and B is the number of integer points on the boundary. Notice that $B \geq 3$ always, since the 3 vertices of the triangle are integer points.

If $I = 1$, what is the largest B can be?

(A) 8 (B) 9 (C) 10 (D) 12 (E) more than 12

Problem 25. Let R_0 be the positive x-axis and $P_0 = (1, 0)$. Given R_n and P_n, let R_{n+1} be the ray in the first quadrant which bisects the angle between R_n and the positive y-axis, and let P_{n+1} be the intersection of R_{n+1} with the line through P_n perpendicular to R_n. The sequence of points $P_0, P_1, \dots$ approaches the y-axis. What is the y coordinate of the limit of that sequence?



(A) $\frac{1+\sqrt{5}}{2}$ (B) 2 (C) e (D) π (E) $\frac{\pi}{2}$