Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

November 8, 2008

Dedicated to the memory of Steve Sigur (Paideia School)

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit quietly.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

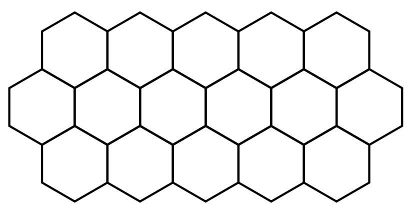
Problem 1. What is the closest integer to

$$\sqrt{9! \cdot 9} + \sqrt{9!/9},$$

where $9! = 9 \cdot 8 \cdot 7 \cdots 1$ denotes "9 factorial"?

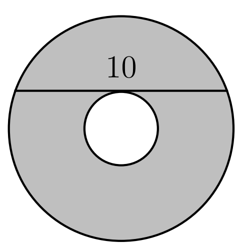
(A) 0 (B) π (C) 9 (D) 2008 (E) 10^6

Problem 2. How many ways are there to color the hexagonal regions of the diagram below with the three colors red, green, and blue so that no two adjacent regions are colored the same?



(A) 0 (B) 3 (C) 6 (D) 12 (E) 16

Problem 3. A chord of a circle is tangent to a smaller, concentric circle. Given that the length of the chord is 10, find the area of the donut shape ("annulus") in between the two circles.



(A) 9π (B) 16π (C) 25π (D) 50π (E) None of the above

Problem 4. How many equilateral triangles can be formed from the vertices of a cube?

(A) 2 (B) 6 (C) 8 (D) 12 (E) 24

Problem 5. What is x if $x \geq 1$ and

$$x^{\log_2 x} = 16,$$

where $\log_2 x$ denotes the logarithm of x to the base 2?

(A) 2 (B) 4 (C) 6 (D) 8 (E) None of the above

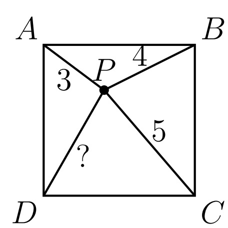
Problem 6. Four pencils are labeled with the names of the four people on a complete UGA math tournament team (all the names are different). How many ways are there to distribute each pencil to a team member so that nobody has their matching pencil? Note that a team member may be given none, one, some, or all of the pencils.

(A) 9 (B) 16 (C) 27 (D) 81 (E) None of the above

Problem 7. Boris has 20 stones in a single pile, and he is trying to split them up so that each stone ends up in a pile by itself. Every time he splits a pile into two new sub-piles, one of size x and the other of size y, he gets $x \cdot y$ points added to his "score". If Boris's initial score is 0, what's the largest final score he can attain?

(A) 190 (B) 200 (C) 210 (D) 300 (E) 400

Problem 8. A point P is chosen inside a square $ABCD$ so that $AP = 3$, $BP = 4$, and $CP = 5$. Find DP.



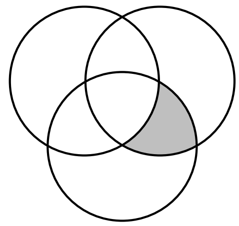
Note: this diagram is not to scale.

(A) 2 (B) 3 (C) 4 (D) 5 (E) None of the above

Problem 9. How many digits are there in the first positive multiple of 6 that contains only the digits 0 and 1 (in its base 10 representation)?

(A) 3 (B) 4 (C) 5 (D) 6 (E) None of the above

Problem 10. In the diagram below, the three circles have unit radius and pass through one another's centers. Find the area of the shaded region.



(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}-\frac{\sqrt{3}}{4}$ (C) $\frac{\pi}{3}-\sqrt{3}$ (D) $\frac{\sqrt{3}}{4}$ (E) None of the above

Problem 11. How many non-negative integers less than 1000 can be expressed as

$$\lfloor x\rfloor+\lfloor 2x\rfloor+\lfloor 5x\rfloor$$

for some real value of x, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x (sometimes written $[x]$ instead)?

(A) 500 (B) 545 (C) 600 (D) 750 (E) 1000

Problem 12. What are the last two digits of

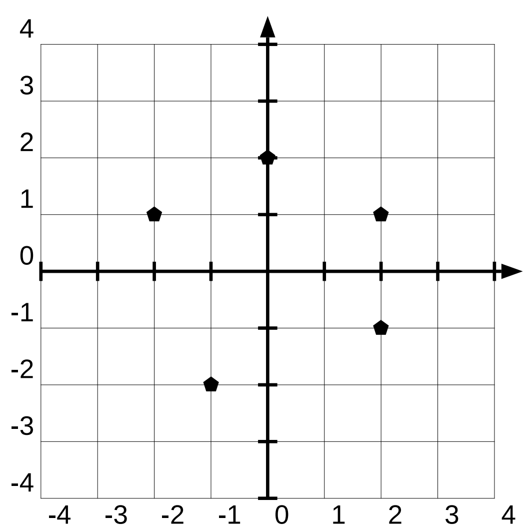
$$2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}}\ ?$$

Recall that x^{y^z} means $x^{(y^z)}$. Therefore, if $f(1)=2$ and $f(n)=2^{f(n-1)}$ for $n>1$, the number above is $f(10)$.

(A) 16 (B) 36 (C) 56 (D) 76 (E) 96

Problem 13. The midpoints of the sides of a (not necessarily convex) pentagon are, in order,

$$(2,1)\quad(2,-1)\quad(-1,-2)\quad(-2,1)\quad(0,2)$$



Which of the following was a vertex of the pentagon? (The midpoints are marked in the grid above.)

(A) (0,0) (B) (0,1) (C) (1,0) (D) (1,1) (E) (1,2)

Problem 14. Evaluate

$$\sqrt{33 \cdot 34 \cdot 35 \cdot 36 + 1}.$$

(A) 1089 (B) 1091 (C) 1190 (D) 1191 (E) None of the above

Problem 15. Suppose

$$\begin{aligned} a + 4b + 9c + 16d + 25e &= -13, \\ 4a + 9b + 16c + 25d + 36e &= -8, \\ 9a + 16b + 25c + 36d + 49e &= 3. \end{aligned}$$

Find

$$16a + 25b + 36c + 49d + 64e.$$

(A) 2 (B) 8 (C) 16 (D) 18 (E) 20

Problem 16. How many of the four 9-digit numbers

111,222,333
111,333,222
222,333,111
333,222,111

are divisible by 13?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 17. Solve for the integer n:

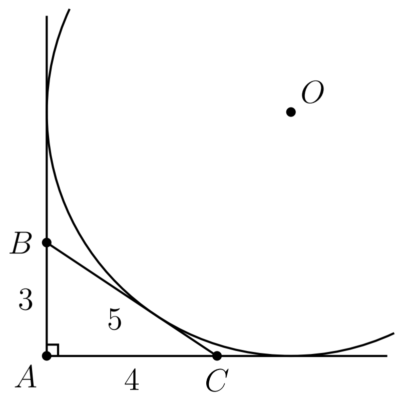
$$3^5 + 54^5 + 62^5 = 24^5 + 28^5 + n^5.$$

(A) 64 (B) 66 (C) 67 (D) 70 (E) 77

Problem 18. If $\cos\theta = \frac{1}{3}$, find $\cos 5\theta$.

(A) $\frac{1}{243}$ (B) $\frac{41}{243}$ (C) $\frac{100}{243}$ (D) $\frac{231}{243}$ (E) $\frac{241}{243}$

Problem 19. In the figure below, the circle is tangent to the hypotenuse and the extensions of the two legs of a 3–4–5 right triangle. Find the radius of the circle.



Note: this diagram is not to scale.

(A) $2 + 2\sqrt{2}$ (B) 6 (C) 7 (D) 8 (E) None of the above

Problem 20. If three positive integers $a < b < c$ satisfy $a^2 + b^2 = c^2$ (and thus correspond to the lengths of the sides of a right triangle) and a, b, and c share no common factor other than 1, we say that a, b, c forms a *primitive* Pythagorean triple.

For example, 16, 63, 65 is a primitive triple, while 39, 52, 65 and 25, 60, 65 are *not* because they are multiples of the 3, 4, 5 and 5, 12, 13 triples, respectively. What is $b - a$ in the other primitive Pythagorean triple where the hypotenuse c is 65?

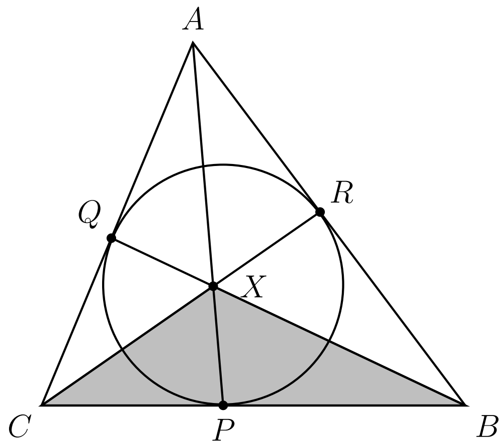
(A) 11 (B) 25 (C) 41 (D) 59 (E) None of the above

Problem 21. Compute

$$\frac{(10^4+2^6)\,(18^4+2^6)\,(26^4+2^6)\,(34^4+2^6)\,(42^4+2^6)}{(6^4+2^6)\,(14^4+2^6)\,(22^4+2^6)\,(30^4+2^6)\,(38^4+2^6)}.$$

(A) 53 (B) 97 (C) 181 (D) 221 (E) None of the above

Problem 22. In triangle ABC below, $CA = 5$, $AB = 6$, and $BC = 7$. The inscribed circle is tangent to the sides BC, AC, and AB at points P, Q, and R, respectively. The lines AP, BQ, and CR concur (all intersect) at the point X. Find the ratio of the area of XBC to the area of ABC.



Note: this diagram is not to scale.

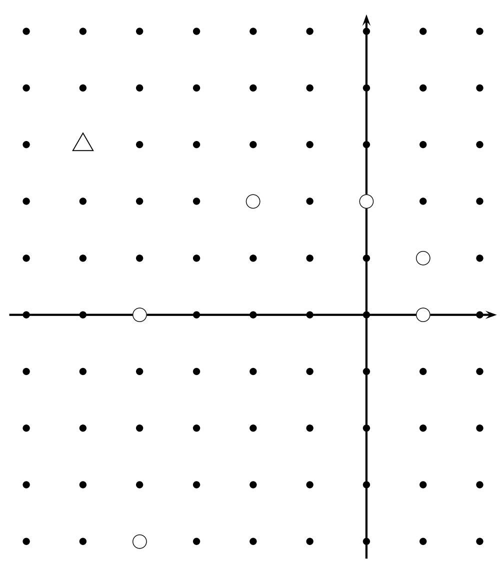
(A) 2/9 (B) 1/4 (C) 7/18 (D) 6/13 (E) None of the above

Problem 23. The complex number $-5+3i$ has a unique representation in base $1+i$, that is, as a sum of powers of $1+i$, some with coefficient 0 and some with coefficient 1. Alternatively, there exists a unique finite set P of non-negative integers so that

$$-5+3i = \sum_{p \in P} (1+i)^p.$$

How many 1-coefficients are in the representation, or alternatively, how many elements are there in P?

For convenience, we have included a scratch complex plane, with the powers of $1+i$ (which are 1, $1+i$, $2i$, $-2+2i$, -4, $-4-4i$, $\ldots$) marked with circles. The target $-5+3i$ is also noted, with a triangle.



(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 24. Suppose that A and B are 3×3 matrices with integral entries and that $AB - BA \equiv I \pmod{3}$, where I is the 3×3 identity matrix. Find $AB^3 - B^3A$ mod 3.

(A) 0 (B) I (C) B (D) B^2 (E) None of the above

Problem 25. When one expands $(x+y)^{2008}$ as

$$1 \cdot x^{2008} + 2008 \cdot x^{2007}y + 2015028 \cdot x^{2006}y^2 + \cdots + 1y^{2008},$$

how many of the coefficients are odd?

(A) 6 (B) 96 (C) 128 (D) 250 (E) 502

Authors. Written by Boris Alexeev with assistance from Valery Alexeev, Mo Hendon, and Ted Shifrin.