Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

## Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

where $C$ is the number of questions answered correctly, $B$ is the number left blank, and $I$ the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit through the rear doors.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

**No calculators are allowed on this test.** 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

**Problem 1.** What is

$$\lceil (6-\sqrt{6})^6 \rceil \ ?$$

(here, $\lceil x \rceil$ denotes the *ceiling* function, i.e. the smallest integer $n$ with $n \geq x$).

(A) 0 (B) $\pi$ (C) 6 (D) 2004 (E) $5^6$

**Problem 2.** Find

$$\frac{6!4!2!}{5!3!1!}$$

(A) 6 (B) 12 (C) 24 (D) 48 (E) None of the above

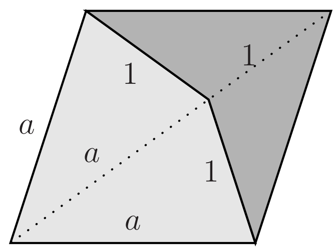
**Problem 3.** In what base $b$, with $b > 7$, is $3 \cdot 7 = 18$?

(A) 8 (B) 10 (C) 12 (D) 14 (E) None of the above

**Problem 4.** What will be the next year when the calendar will be exactly the same as in 2004 (i.e. all the days, January 1 through December 31, will fall on the same days of the week)?

(A) 2005 (B) 2008 (C) 2032 (D) 2040 (E) Never

**Problem 5.** In the picture below, what is $a$? (The large figure is a rhombus.)




(The two tiles pictured are called Penrose tiles, "kite" and "bat".)

(A) $\sqrt{5}-1$ (B) 1 (C) 2 (D) $\dfrac{1+\sqrt{5}}{2}$ (E) None of the above

**Problem 6.** In a $4 \times 4$ magic square, with numbers 1 through 16, what is the sum of the numbers in any row or column? (In a magic square, the numbers in each row, column, and diagonal have the same sum.)

(A) 16 (B) 17 (C) 32 (D) 36 (E) None of the above

**Problem 7.** There are four cards, two red and two black. Two cards are chosen at random. What is the probability that they have the same color?

(A) 1/4 (B) 1/3 (C) 1/2 (D) 2/3 (E) None of the above

**Problem 8.** Call a positive integer *evil* if it is 666 larger than the sum of its digits. How many evil numbers are there?

(A) 0 (B) 9 (C) 10 (D) 18 (E) 666

**Problem 9.** In a Rubik's cube (consisting of $3 \times 3 \times 3 = 27$ smaller cubes) how many diagonals of all kinds there are?

Here, a *diagonal* is defined to be a straight line consisting of 3 distinct cells. For example, in a $3 \times 3$ square there are $3 + 3 + 2 = 8$ diagonals, 3 horizontal, 3 vertical and 2 going from one corner to another.

(A) 27 (B) 35 (C) 49 (D) 76 (E) None of the above

**Problem 10.** You are in charge of designing a new system of coins. You are allowed to design coins with any integral value but executive order demands that any integral amount from 1 to 99 cents be obtainable without using any type of coins twice. What is the smallest number of types of coins that you must have?

(A) 5 (B) 6 (C) 7 (D) 8 (E) More that 8

**Problem 11.** Three circles of radius 1 are centered at the vertices of the 3–4–5 triangle. (When we say "circle" we include the boundary but not the interior.) A point on one of the circles is called *invisible* if it cannot be seen from one of the other circles (one cannot see through the circles). What is the total length of the set of invisible points?

(A) 3 (B) $\pi$ (C) 5 (D) $2\pi$ (E) None of the above

**Problem 12.** Find

$$\sum_{j=0}^{5} \binom{5}{j} (-2)^j$$

(A) $-1$  (B) 0  (C) 1  (D) 243  (E) None of the above

**Problem 13.** Find the minimum value of the following function:

$$f(x) = (x-5)^2 + (x-7)^2 - (x-4)^2 - (x-8)^2 + (x-3)^2 + (x-9)^2$$

(A) 8  (B) 10  (C) 12  (D) 14  (E) None of the above

**Problem 14.** How many different words can you form from the letters ABCD, where a word is a sequence of one to four letters, using every letter at most once (for example, words DC and CADB)?

(A) 24  (B) 48  (C) 64  (D) 72  (E) None of the above

**Problem 15.** The centers of the faces of a regular tetrahedron are connected to form a smaller tetrahedron. What is the ratio of the volumes of the bigger and smaller tetrahedra?

(A) 5  (B) 8  (C) 9  (D) 27  (E) 64

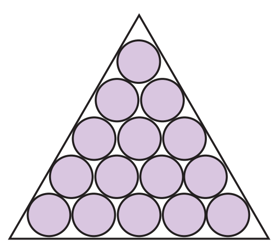
**Problem 16.** In a certain fictional country families stop having children precisely when they produce both a boy and a girl (at least one of each). Assuming boys and girls are equally likely, how many children does an average family have?

(A) 3 (B) 3.5 (C) 4 (D) 5 (E) None of the above

**Problem 17.** A person is standing on a rectangular grid. He is allowed to move one step south, north, east or west. After 4 moves he is supposed to get back to the starting point. How many possibilities are there for the sequence of these 4 moves?

(A) 20 (B) 24 (C) 30 (D) 36 (E) None of the above

**Problem 18.** An equilateral triangle is filled with $n$ rows of (shaded) congruent circles. (The case $n = 5$ is pictured below.) As the number $n$ goes to infinity, the fraction of the triangle that is shaded approaches



(A) $3/4$ (B) $\dfrac{\pi}{2\sqrt{3}}$ (C) $3/\pi$ (D) 1 (E) None of the above

**Problem 19.** Let $f(x) = \{5x/2\}$, the fractional part of $5x/2$. How many solutions in the interval $0 \le x < 1$ does the following equation have?

$$f(f(f(x))) = 0$$

(A) 12 (B) 18 (C) 27 (D) 79 (E) None of the above

**Problem 20.** Arrange $e^\pi$, $3^{\sqrt{5}}$ and $\pi^e$ in increasing order.

(A) $e^\pi < 3^{\sqrt{5}} < \pi^e$ (B) $3^{\sqrt{5}} < \pi^e < e^\pi$ (C) $\pi^e < 3^{\sqrt{5}} < e^\pi$ (D) $3^{\sqrt{5}} < e^\pi < \pi^e$ (E) $e^\pi < \pi^e < 3^{\sqrt{5}}$

**Problem 21.** What is the largest area of an ellipse that can be inscribed in a triangle with sides 3, 4 and 5?

(A) $\pi$ (B) $\pi\sqrt{2}/2$ (C) $\sqrt{3}\pi/2$ (D) $2\pi/\sqrt{3}$ (E) None of the above

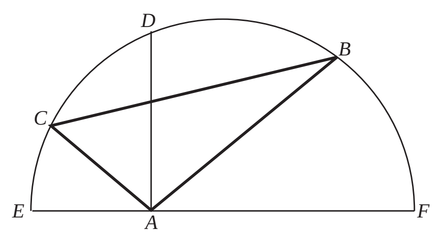
**Problem 22.** There are 3 light switches turned off. Every minute a person comes by and randomly flips exactly one of the switches. On average, how many minutes will pass before all three lights first get turned on at the same time?

(A) 3 (B) 8 (C) 10 (D) 12 (E) None of the above

**Problem 23.** If you interchange the hour and the minute hands on the watch, how many times during the 24-hour day will you still get a legal time? (For example, this is true at noon but not true at 6 a.m.).

(A) 143  (B) 144  (C) 264  (D) 286  (E) 288

**Problem 24.** Given a semicircle with diameter $\overline{EF}$ as indicated, triangle $\triangle ABC$ with $A$ lying on the diameter, and $B$ and $C$ on the semicircle. $\overline{AD}$ bisects $\angle BAC$, $\overline{BE}$ bisects $\angle ABC$, and $\overline{CF}$ bisects $\angle ACB$. If $AB = 6$, $AC = 3$, $BC = 7$, and $\overline{AD} \perp \overline{EF}$, find $AD$.




(A) 4  (B) $3\sqrt{2}$  (C) $9/2$  (D) $3\sqrt{5}$  (E) None of the above

**Problem 25.** Alice, Bob, Caroline, Dave and Emily compete to see who can solve the most problems on this test. In how many different orders can they finish if ties can happen? (For example, for two people there are 3 possible outcomes, and for three, 13 outcomes.)

(A) 478  (B) 480  (C) 541  (D) 600  (E) None of the above

**Authors.** Written by Valery Alexeev and Boris Alexeev with assistance by Ted Shifrin ©2004.