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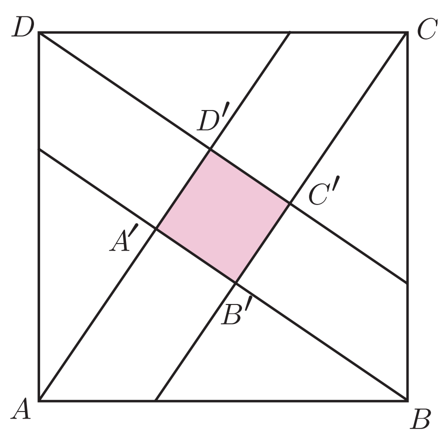
Written test, 25 problems / 90 minutes

WITH SOLUTIONS

1 Easy Problems

Problem 1. On the picture below (not to scale, adapted from an actual Chinese drawing from 1000 B.C.) the area of the large square $ABCD$ is 25, and the area of the small square $A'B'C'D'$ is 1. Find the length of AA'.



(A) 1 (B) 2 (C)$^\heartsuit$ 3 (D) 4 (E) None of the above

Solution. The side of the large square is 5, and that of the small square is 1. Denote AA' by x. The triangle ABA' is a right triangle; so by the Pythagorean theorem we have

$$x^2 + (x+1)^2 = 5^2$$

Then $x = 3$ is an obvious solution (the other solution of this quadratic equation is negative).

(By the way, this Chinese drawing is claimed to represent the proof of the Pythagorean theorem dating many centruries before Pythagoras was born. Can you see the proof? This is not part of the problem.)

Problem 2. Of the first 3,000,000,000 positive integers, what portion is divisible by 2 but not by 3?

(A) 1/6 (B)$^\heartsuit$ 1/3 (C) 1/2 (D) 2/3 (E) None of the above

Solution. Half of these integers are divisible by 2 and two thirds of them are not divisible by 3. So the answer is

$$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

.

Problem 3. How many two-digit numbers double when the two digits are interchanged?

(A) 1 (B) 2 (C) 3 (D) 4 (E)$^\heartsuit$ none

Solution. Let $x = 10a + b$ be such a number. Then

$$10b + a = 2(10a + b), \text{ i.e. } 19a = 9b$$

This means that the digit b has to be divisible by 19, which is impossible.

Problem 4. Let x be the smallest positive integer which gives remainder 1 when divided by 2, remainder 2 when divided by 3, remainder 3 when divided by 4 and remainder 4 when divided by 5. What is the sum of the digits of x?

(A) 11 (B) 12 (C) 13 (D)$^\heartsuit$ 14 (E) 15

Solution. Clearly, $x+1$ is the smallest positive integer which is divisible by 2,3,4 and 5. Hence,

$$x+1 = 3 \times 4 \times 5 = 60, \quad \text{and} \quad x = 59$$

The sum of digits of x is $5+9=14$.

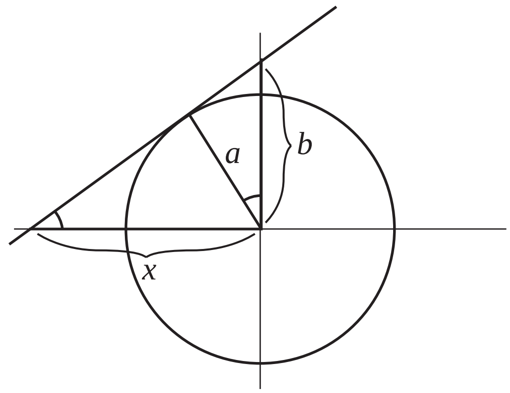
Problem 5. If the line $y = mx + b$ with $b > 0$ is tangent to the circle $x^2 + y^2 = a^2$, then

(A) $b = a^2(m^2+1)$ (B) $a = b\sqrt{m^2+1}$ (C) $ab = m^2+1$
(D)$^\heartsuit$ $b^2 = a^2(m^2+1)$ (E) not enough information

Solution. By similar triangles, we have

$$\frac{a}{b} = \frac{x}{\sqrt{x^2+b^2}} = \frac{x}{x\sqrt{m^2+1}} = \frac{1}{\sqrt{m^2+1}}\,.$$

Therefore, $b = a\sqrt{m^2+1}$.



Problem 6. Suppose you are writing positive integers in a row, without blank spaces, like this:

$$123456789101112\ldots$$

What will be the 1000th digit?

(A) 1 or 2 (B)$^\heartsuit$ 3 or 4 (C) 5 or 6 (D) 7 or 8 (E) 9 or 0

Solution. The one-digit numbers $1-9$ will take up 9 digits, and the two digit numbers $10-99$ will take up $90\times 2 = 180$ digits. This leaves $1000-9-180 = 811$ digits for the three-digit numbers. We have $811 = 270 \times 3 + 1$, so the 1000th digit will be the first digit from the left in the 271st three-digit number. That number is $100 + 271 - 1 = 370$, and the digit is 3.

Problem 7. Find

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + 30^2$$

(A) 465 (B) 7855 (C) 9402 (D)$^\heartsuit$ 9455 (E) 13505

Answer. 9455

Solution. Of course, if you know the formula

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6}$$

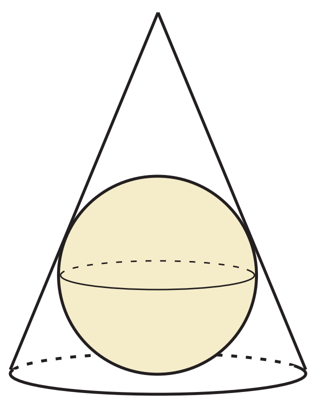
then you just plug in $n = 30$ to obtain 9455. Even if you do not remember it, you can estimate the sum as

$$\int_0^n x^2 dx = \frac{n^3}{3},$$

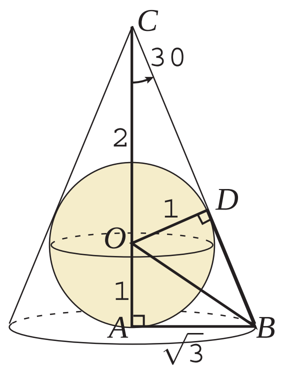
which for $n = 30$ gives 9000. You can also notice that it must be an odd number. That leaves only one choice.

Problem 8. A sphere is inscribed in a right circular cone with vertex angle $60°$. The ratio of the volume of the sphere to the volume of the cone is

(A) $1/3$ (B) $\sqrt{3}/4$ (C)$^\heartsuit$ $4/9$ (D) $\sqrt{2}/3$ (E) $1/2$



Solution. Say the sphere has radius 1. Let D be the point of contact of the sphere and cone, as pictured.



Then OB bisects $\angle CBA = 60°$, and we have $\angle ACB = \angle OBA = 30°$. Therefore, $OC = 2$ and $AB = \sqrt{3}$, so

$$\frac{\text{volume sphere}}{\text{volume cone}} = \frac{\frac{4}{3}\pi}{\frac{1}{3}\pi(\sqrt{3})^2(3)} = \frac{4}{9}.$$

Problem 9. Alice, Bob, Charlie, Diane and Ed sit at a round table in random order. What is the probability that Alice and Bob are neighbors?

(A) 1/8 (B) 1/4 (C) 1/6 (D)$^{\heartsuit}$ 1/2 (E) 2/3

Solution. Fix Ed. Then we get the problem from the ciphering round about A,B,C and D in a line. So the probability is 1/2, as it was there.

Alternatively, fix Alice. Of all four places Bob can sit, two are adjacent to Alice, so there is a probability of $2/4 = 1/2$ that Bob sits next to Alice.

Problem 10. If $aaa_9 = bbb_{16}$ (the first numeral is in base 9 and the second one is in base 16) then $a/b =$

(A) 1 (B) 2 (C)$^\heartsuit$ 3 (D) 4 (E) None of the above

Solution. $a(1+9+9^2) = b(1+16+16^2)$, so $91a = 273b$, so $a/b = 273/91 = 3$.

Problem 11. Find $\sqrt{x^2-y^2}$ if x and y satisfy the following system of equations:

$$\begin{aligned} x+y+\sqrt{x+y} &= 72 \\ x-y-\sqrt{x-y} &= 30 \end{aligned}$$

(A) 24 (B) 30 (C)$^\heartsuit$ 48 (D) 72 (E) None of the above

Solution. The first equation is quadratic in $\sqrt{x+y}$. Solving it, we get $\sqrt{x+y} = 8$ (or -9, which is impossible). Likewise, the second equation gives $\sqrt{x-y} = 6$ (or -5). Therefore,

$$\sqrt{x^2-y^2} = \sqrt{x+y}\cdot\sqrt{x-y} = 8 \cdot 6 = 48.$$

2 Medium Problems

Problem 12. Among the following shapes of equal area, which one has the largest perimeter?

(A) circle (B)$^\heartsuit$ triangle (C) square (D) regular pentagon
(E) regular hexagon

Solution. It is widely known that a circle is "optimal", in the sense that it has the largest area for a given perimeter; hence the smallest perimeter for a given area. The closer a polygon is to the circle, the more "optimal" it is. Hence, the triangle is the least "optimal", and has the largest perimeter.

Problem 13. Only one of the following numbers is prime. Which one?

(A)$^\heartsuit$ 19972003 (B) 19992003 (C) 20012003 (D) 20022003
(E) 20032003

Solution. 19992003 and 20022003 are divisible by 3, by applying the divisibility criterion: the sum of the digits is divisible by 3. 20012003 is divisible by 11: the alternating sum of the digits is divisible by 11. 20032003 is obviously divisible by 2003. This leaves only 19972003.

Problem 14. Let us play the following game. You have \$1. With every move, you can either double your money or add \$1 to it. What is the smallest number of moves you have to make to get to \$200?

(A) 6 (B) 7 (C) 8 (D)$^\heartsuit$ 9 (E) It is impossible to get to \$200

Solution. Write 200 in the binary system: 11001000. Starting with 1, with every move you can either write a 0 at the end, or add 1 — and if the last binary digit is 0, that will make it into 1. Clearly, you can get to any binary number in this way.

Now, look at the following quantity: the number of digits + the number of 1's. With every "good" move as above, this quantity increases by one. With every "bad" move (adding 1 when the last digit is 1), this quantity does not increase by more than one. Therefore, the minimal number of moves is $(8+3)-(1+1)=9$.

Problem 15. You repeatedly throw a coin. What is the probability that heads comes up three times before tails comes up twice?

(A) 1/16 (B) 3/16 (C)$^\heartsuit$ 5/16 (D) 1/2 (E) None of the above

Solution. Everything will be decided after we know the results of the first

4 throws. Of the $2^4 = 16$ possibilities, there are 5 that satisfy our condition: HHHH, HHHT, HHTH, HTHH, THHH.

Problem 16. A ball is shot from a corner of a square billiard table with a side 1. It bounces 3 times off the walls and then falls into a corner. What is the greatest distance it could have possibly traveled?

(A) $\sqrt{13}$ (B) 4 (C)$^\heartsuit$ $\sqrt{17}$ (D) 5 (E) None of the above

Solution. Draw the rectangular grid with lines at distance 1 from each other. The path of a billiard ball corresponds to a straight line from point $(0,0)$ to point (m,n) with positive integral m and n so that:

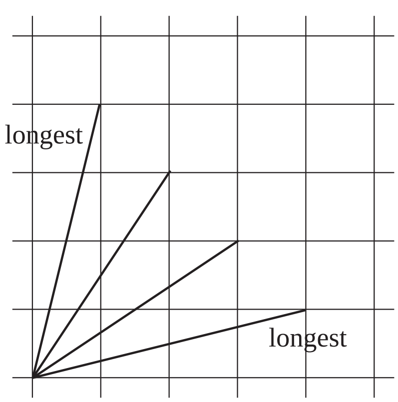
1. It does not pass through other points with integral coefficients — that is equivalent to requiring that m,n are relatively prime.

2. It crosses the lines 3 times — that is equivalent to the condition $(m-1)+(n-1)=3$, i.e. $m+n=5$.

One has the following possibilities for m and n:

$$m+n = 4+1 = 3+2 = 2+3 = 1+4$$

The maximal distance is $\sqrt{4^2+1^2} = \sqrt{17}$.



Problem 17. If $a + b = 1$ and $a^3 + b^3 = 4$, then $a^4 + b^4 =$

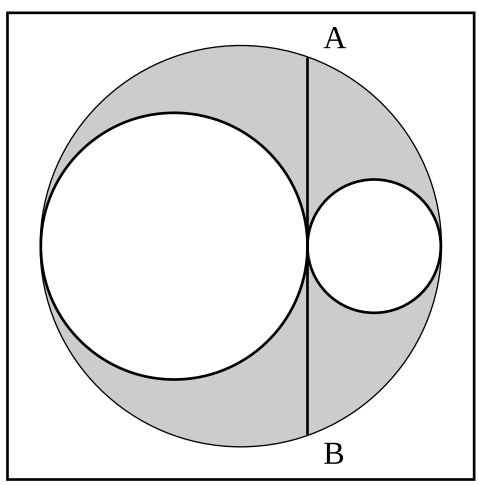
(A) 1 (B) 3 (C)$^\heartsuit$ 7 (D) 9 (E) none of the above

Solution. Since $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$, we conclude that $ab = -1$. Therefore,

$$\begin{aligned} a^4 + b^4 &= (a + b)^4 - (4a^3b + 6a^2b^2 + 4ab^3) \\ &= (a + b)^4 - ab\big(4(a + b)^2 - 2ab\big) \\ &= 1 + (4 + 2) = 7\,. \end{aligned}$$

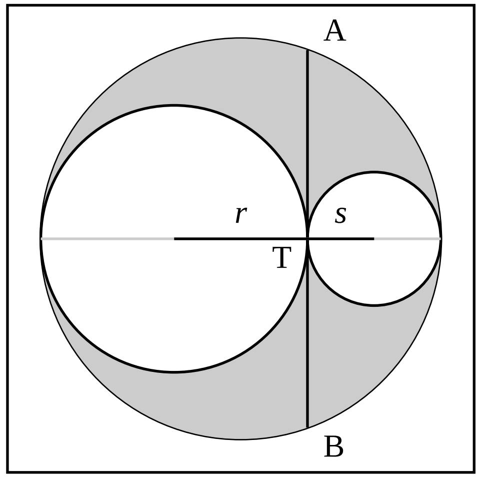
(Alternatively, once we know $ab = -1$, from $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ we conclude that $a^2 + b^2 = 3$, and so $a^4 + b^4 = (a + b)(a^3 + b^3) - ab(a^2 + b^2) = 4 + 3 = 7$.)

Problem 18. The length of the chord AB is 4. Find the area of the shaded region.



(A) $\pi/2$ (B) π (C) $\pi + 1$ (D)$^\heartsuit$ 2π (E) None of the above

Solution. Let the radii of the two small circles be r and s. Then the radius of the large circle is $r + s$.



Computing the power of point T,

$$AT \cdot TB = 2 \cdot 2 = 2r \cdot 2s \qquad \Rightarrow \qquad rs = 1$$

We wish to find

$$\pi \left((r+s)^2 - r^2 - s^2 \right) = 2\pi rs = 2\pi.$$

(Alternatively, drawing the radius of the large circle to A, the Pythagorean Theorem gives $(r-s)^2 + 4 = (r+s)^2$, so $rs = 1$.)

Problem 19. There are 120 permutations of the word **BORIS**. Suppose these are arranged in alphabetical order, from BIORS to SROIB. What will be the 60th permutation?

(A) ORSIB (B) OSBIR (C)♡ OISRB (D) OBSIR (E) OIBRS

Solution.

$$60 = 2 \cdot 24 + 2 \cdot 6 = 2 \cdot 4! + 2 \cdot 3!$$

This means that by the 60th permutation we will go through all 4! combinations starting with letter B, and 4! combinations starting with letter I. Hence, the first letter will be O.

Likewise, for the remaining letters BIRS, we will go through all combinations starting with B, and the 60th permutation will be the last combination starting with I, that is ISRB. So, the combination we are looking for is OISRB.

Problem 20. How many times during a 24-hour day are the hour hand and the minute hand of a watch perpendicular to each other? (For example, this is true at 3 a.m.).

(A) 4 (B) 22 (C) 24 (D)$^\heartsuit$ 44 (E) 48

Solution. We will find the number for the period of 12 hours and then multiply the answer by two to get the answer for the 24-hour day.

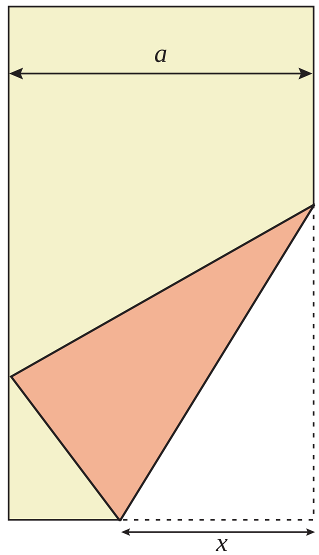
Let $0 \le x < 12$ represent the hours (it need not be integral, e.g., 1 hour 25 minutes means that $x = 1\ 25/60$). Then the minute hand points at $y = 12x \mod 12$. Our condition says that

$$\begin{aligned} 12x &= x + 3 \mod 12 \quad \text{or} \quad 12x = x - 3 \mod 12 \\ 11x &= 3 \mod 12 \quad \text{or} \quad 11x = -3 \mod 12 \end{aligned}$$

Each of these equations has exactly 11 solutions separated by 12/11. So, there are 22 occurrences during the 12 hours, and 44 during the day.

3 Hard Problems

Problem 21. A (very long) piece of paper is folded, as pictured, bringing the right bottom corner to the left edge of the paper. If the width of the paper is a, and the length folded over is x, as marked in the picture, then the length of the crease is

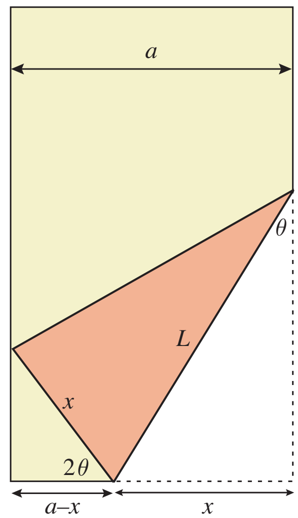


(A) $\dfrac{ax\sqrt{2}}{2x-a}$ (B) $x\sqrt{2}$ (C) $\dfrac{x\sqrt{2a}}{\sqrt{2a-x}}$

(D) $x^2\sqrt{6}\sqrt{4x^2-a^2}$ (E)$^{\heartsuit}$ $\sqrt{\dfrac{2x^3}{2x-a}}$

Solution. As pictured, $\sin\theta = \dfrac{x}{L}$ and $\cos 2\theta = \dfrac{a-x}{x} = \dfrac{a}{x} - 1$. From $\cos 2\theta = 1 - 2\sin^2\theta$ we obtain

$$\frac{a}{x} - 1 = 1 - 2\frac{x^2}{L^2}\,, \qquad \text{and so} \quad L^2 = \frac{2x^3}{2x-a}\,.$$



Problem 22. Among the numbers 2000, 2001, 2002, 2003, how many can be written in the form $n^2 + m^2$ for some integers n and m?

(A) 0 (B)$^\heartsuit$ 1 (C) 2 (D) 3 (E) 4

Solution. $2000 = 1600 + 400 = 40^2 + 20^2$. The other numbers cannot be written as sums of two squares. For 2003, look at squares modulo 4. A square n^2 modulo 4 is either 0 or 1. Hence, $n^2 + m^2$ is either 0 (and then it has to be divisible by 4) or 1 or 2. But 2003 equals 3 modulo 4.

For 2001, do the same modulo 3. A square n^2 equals either 0 or 1 modulo 3. So, $n^2 + m^2$ is either 0 (and then it is divisible by 9) or 1 or 2. The number 2001 equals 0 modulo 3 but is not divisible by 9.

The number 2002 is the hardest, but the same argument works if we do it modulo 11.

Problem 23. What is the number of pairs (x, y) of integers satisfying

$$x^2 + y^2 \leq 100\,?$$

(A) 101 (B) 179 (C) 297 (D)$^\heartsuit$ 317 (E) 361

Solution. We are looking for the number of points (x, y) with integer coordinates inside a circle of radius 10. This is approximately the area of the circle. Hence, we expect to get approximately $\pi 10^2 \approx 314$ pairs. The number 317 is by far the closest to this.

Problem 24. Find out how many numbers in the 100th row of the Pascal triangle (the one starting with 1, 100, ...) are not divisible by 3.

(A) 4 (B)$^\heartsuit$ 12 (C) 27 (D) 32 (E) None of the above

Solution. We need to find the number of coefficients in the polynomial

$$(1+x)^{100} = 1 + \binom{100}{1} x + \binom{100}{2} x^2 + \cdots + x^{100}$$

which are not equal to 0 modulo 3. Note that modulo 3 one has

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3 \equiv 1 + x^3$$

(this is called Freshman's Dream sometimes, or the high school student's binomial theorem), and so also

$$(1+x)^9 \equiv (1+x^3)^3 \equiv 1 + x^9,$$

etc, for any power of 3. Now, $100 = 81 + 2 \cdot 9 + 1$. Therefore, modulo 3 one has

$$(1+x)^{100} = (1+x)^{81} \left((1+x)^9\right)^2 (1+x) = (1+x^{81})(1+2x^9+x^{18})(1+x)$$

In this product all $2 \cdot 3 \cdot 2 = 12$ powers of x are different (because every integer can be written in base 3 in a unique way), and the coefficients are all nonzero modulo 3. So, the answer is 12.

Note that we basically proved a very general theorem (due to Gauss): if n is written as $a_k a_{k-1} \dots a_0$ in base p, where p is a prime number, then in the n-th row of Pascal triangle there are

$$(a_k + 1)(a_{k-1} + 1) \dots (a_0 + 1)$$

coefficients which are not divisible by p.

Problem 25. Among the first one billion positive integers, consider the sets of:

(1) palindromic numbers (such as 22, 121, 11533511, etc.),

(2) prime numbers (such as 2, 3, 5, 7, etc.), or

(3) perfect cubes (such as 1, 8, 27, 64, etc.).

Arrange these in the order of decreasing size.

(A) 1,2,3 (i.e. palindromic numbers are the most frequent, then primes, then cubes) (B) 1,3,2 (C)$^\heartsuit$ 2,1,3 (D) 2,3,1 (E) 3,1,2

Solution. Let us estimate how many of these types of numbers we have which are $\leq n$ for some large n. For palindromic numbers, it is about $\sqrt{n}$ since only the last half of the digits matter. For perfect cubes, the answer is certainly $\sqrt[3]{n}$. For prime numbers, the answer is known to be $n/\ln n$, from number theory. Since $\ln n \ll n^c$ for any $c > 0$ (for large n), we have

$$\frac{n}{\ln n} \gg \sqrt{n} \gg \sqrt[3]{n}$$

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