Real Analysis Qualifying Exam

January, 2025

State clearly which theorem you are using. Conclusions in one problem in this exam may be used for another without proof. Each problem is worth 10 points.

Notation: m is the Lebesgue measure on the space $\mathbb{R}^n$, and so is dx.

1. Let f be a continuous complex valued function on $[0,1]$. Show that

$$\|f\|_\infty = \sup\{|\lambda| : \lambda \in \mathbb{C}, m(f^{-1}(B(\lambda, \varepsilon))) > 0 \text{ if } \varepsilon > 0\},$$

   where $B(\lambda, \varepsilon) \subset \mathbb{C}$ is the open disc of radius ε centered at λ.

2. Show that Lebesgue measurable functions f_n defined on $[a,b]$ converge in measure to f if and only if

$$\lim_{n\to\infty} \int_a^b \frac{|f_n - f|}{1 + |f_n - f|}\, dx = 0.$$

3. Show that if f is Riemann integrable on $[a,b]$ and $f(x) = 0$ for $x \in \mathbb{Q} \cap [a,b]$, then $\int_a^b f(x)dx = 0$.

4. Let $f_n \in L^2([0,1])$ be a sequence that is bounded in norm $\|\cdot\|_2$ and $f_n \to 0$ a.e. Show that $\lim\limits_{n\to\infty} \|f_n\|_1 = 0$.

   Hint: Use Egoroff and Cauchy-Schwarz.

5. Show that for every $f \in L^1(\mathbb{R})$, there exists a sequence of continuous functions g_n with compact supports such that $\lim\limits_{n\to\infty} \|g_n - f\|_1 = 0$.

6. Let $f \in L^1[0,1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)\,|\sin nx|\, dx = \frac{2}{\pi}\int_0^1 f(x)dx.$$

   Hint: start with the easiest function f first.