Topology Qualifying Exam Spring 2025

January 2, 2025

Each problem is worth 10 points. There are 9 problems, but you only need to do 8 of them, see instructions below.

Point-set topology: Do two of the following three problems.

1. Suppose that X is a compact Hausdorff topological space and $A \subset X$ is closed. Prove that X/A is compact and Hausdorff.

2. Let X be a complete nonempty metric space. Let $f : X \to X$ a continuous map such that there is a real number $r \in (0, 1)$ with

   $$d(f(x), f(y)) \leq r\, d(x, y)$$

   for all $x, y \in X$. Prove that f has a unique fixed point.

3. Let $\{X_\alpha : \alpha \in A\}$ be a (possibly infinite) family of subsets of a topological space X. Suppose that

   $$\cap_{\alpha \in A} X_\alpha$$

   is nonempty and that each X_α is connected. Prove that

   $$\cup_{\alpha \in A} X_\alpha$$

   is connected. (Warning, "connected" is not the same as "path-connected", and X and the X_α's may not be path-connected.)

Algebraic topology: Do all of the following six problems. Here you may use as given the fundamental group and homology groups of points and spheres; for all other spaces, if you need to know these groups then you should explain your computations.

4. Prove that every contractible space is simply connected. Give an example of a simply connected space which is not contractible.

5. Prove that any map $\mathbb{R}P^2 \to S^1 \times S^1$ is null-homotopic. Prove that there exists a map $S^1 \times S^1 \to \mathbb{R}P^2$ which is not null-homotopic.

6. Let Σ_g denote a closed orientable surface of genus g, and let $\Sigma_{g,k}$ be a compact orientable surface with $k \geq 0$ boundary components (i.e. Σ_g with k open disks removed). For which g and k is there a continuous map $f : \Sigma_{g,k} \to \Sigma_{g,k}$ which is homotopic to the identity but does not have any fixed points?

7. Let A and B be copies of the solid torus $S^1 \times B^2$, with coordinates $\{\psi\} \times \{(r, \theta)\}$, where ψ is the angular coordinate on S^1 and (r, θ) are standard polar coordinates on $B^2 \subset \mathbb{R}^2$. Let $f : \partial A \to \partial B$ be the map from the boundary of A to the boundary of B defined by:

$$f(\psi, 1, \theta) = (\psi + 2\theta, 1, \psi + \theta)$$

   Let X be the result of gluing A to B using the map f, i.e.

$$X = A \amalg B/p \sim f(p)$$

   Use the Seifert-Van Kampen theorem to compute $\pi_1(X)$.

8. Given any topological space X, let the *suspension* $S(X)$ of X be the quotient space of $X \times [0, 1]$ in which $X \times \{0\}$ is collapsed to a single point S and $X \times \{1\}$ is collapsed to another point N. Draw a picture to illustrate this construction and then use a Mayer-Vietoris sequence to find a relationship between the integral homology groups of X and the integral homology groups of $S(X)$.

9. Let $X = Y \cup Z \subset \mathbb{R}^3$, where $Y = \{x^2 + y^2 + z^2 = 1\}$ is the unit sphere and $Z = \{x^2 + y^2 + \frac{z^2}{4} = 1\}$ is an ellipsoid. Draw a picture of X, explicitly describe a CW-complex structure on X, write down the cellular chain complex associated to this CW-complex structure, and use this to compute the integral homology of X.