Qualifying Examination Problems, Spring, 2025

There are 8 problems from three subjects. If you complete 5 of them correctly, you have a strong pass of the examination. If you complete 4 of them correctly and some partial credits with total points more than 50 points over the 80 points, you get a master pass.

Numerical Linear Algebra

- 1.(10 points) Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be a set of vectors in $\mathbb{R}^n$ and A be a symmetric positive definite matrix of size $n \times n$.

  (3 points) Explain the condition that these vectors are A-orthogonal.

  (3 points) To solve $A\mathbf{x} = \mathbf{b}$ using these A-orthogonal vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ by the following iterations:

  $$\mathbf{x}_k = \mathbf{x}_{k-1} + t_k \mathbf{v}_k$$

  for $k = 1, \cdots, n$ with $\mathbf{x}_0$ being a given initial vector. Here t_k is chosen so that $q(\mathbf{x}_k) = \langle \mathbf{x}_k, A\mathbf{x}_k \rangle - 2\langle \mathbf{x}_k, \mathbf{b} \rangle$ is minimized. Find the formula for t_k.

  (4 points) Show that $A\mathbf{x}_n = \mathbf{b}$.

- 2(10 points) Explain Jacobi iterative method to solve linear system of equations $A\mathbf{x} = \mathbf{b}$ and show the convergence when matrix A is strictly diagonally dominant.

  (3 points) Derive this method in detail.

  (2 points) Give a definition of strictly diagonally dominant matrix.

  (5 points) Show that Jacobi iterative method converges when matrix A is strictly diagonally dominant.

- 3(10 points) Let A be an invertible $n \times n$ matrix. Consider the Gaussian elimination method to solve $A\mathbf{x} = \mathbf{b}$ with pivoting.

  (5 points) Formulate the method, and explain why pivoting is sometimes necessary.

  (5 points) Explain how the Gaussian elimination method with pivoting applied to A leads to the decomposition $PA = LU$ where P is a permutation matrix, L is lower-triangular and U is upper-triangular.

Numerical Approximation

- 4(10 points) Given distinct points $x_0, \ldots, x_n \in \mathbb{R}$, and a function f defined on $\mathbb{R}$ with sufficient regularity.

  (3 points) State the definition of the Hermite interpolation of f at $x_0, \ldots, x_n$, if one requires the interpolation to match the derivative of f up to order 1 at each x_i .

  (3 points) Give an explicit formula for the Hermite interpolation in the Lagrange form, i.e., find $A_i(x), B_i(x)$ such that the Hermite interpolation of f is

  $$p(x) = \sum_{i=0}^{n} f(x_i)A_i(x) + \sum_{i=0}^{n} f'(x_i)B_i(x)$$

  (4 points) Prove the uniqueness of this Hermite interpolation (i.e., prove the uniqueness of a polynomial satisfying the characterizing properties of this Hermite interpolation).

- 5(10 points) Consider the composite trapezoid rule for the integral $\int_a^b f(x)dx$.

  (3 points) Formulate the method when the interval $[a, b]$ is cut into n equal pieces.

  (7 points) Assume $f \in C^2[a, b]$. Show that its error is

  $$-\frac{1}{12}(b-a)h^2 f''(\xi)$$

  where $h = \frac{b-a}{n}$, $\xi \in (a, b)$.

- 6(10 points) Consider the numerical approximation of $f'(x)$ using its point values at equally spaced grid points.

  (6 points) Derive such an approximation whose accuracy order is at least 3. The accuracy order should be justified.

  (4 points) Apply the Richardson extrapolation to improve the accuracy order of the method you derived.

Numerical Differential Equations

- 7(10 points) Consider the ODE

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0$$

  (4 points) Write the formula for the modified Euler method (also known as the midpoint method) and write its Butcher table.

  (6 points) Show that the modified Euler method is at least second order accurate.

- 8(10 points) Consider the second order ODE

$$x''(t) = x'(t) + (x(t))^2, \quad x(0) = 1, \quad x'(0) = -2$$

  (3 points) Reformulate it as an initial value problem of a system of first order ODEs.

  (4 points) Apply a second order multistep method to solve this system. Give the explicit formulas of the $(i+1)$-th step in terms of the i-th step and the $(i-1)$-th step.

  (3 points) Prove that the multistep method you used is at least second order.