Complex Analysis Qualifying Exam – Spring 2025

All problems are of equal weight. Please arrange your solutions in numerical order even if you do not solve them in that order. Show work and carefully justify/prove your assertions.

1. (10 points)
   (a) (7 points) If f is analytic and not a constant in a domain D, show that $|f|$ cannot have in D a local maximum.
   (b) (3 points) If f is analytic and not a constant in a domain D, show that $|f|$ cannot have a local minimum at $z_0 \in D$, unless $f(z_0) = 0$.
   [Hint (b): use part (a).]

2. (10 points) Compute
$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)\cosh(\pi x/2)}\, dx.$$
   [Hint: It is convenient to compute an integral along the square with vertices $-N$, N, $-N+2iN$ and $N+2iN$. Use the fact that
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \log 2.]$$

3. (10 points) Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$, and $a \neq 0$.
   [Hint: Consider singularity at 0 of a function $z \mapsto f(1/z)$.]

4. (10 points) Show that all zeros of the polynomial $P(z) = z^5 - z + 16$ are contained in the annulus $1 < |z| < 2$.

5. Let $f(z)$ be a holomorphic function in a neighborhood of $z = 0$ which satisfies the functional equation
$$f(z) = z + f(z^2).$$
   (a) (5 points) (10 points) Find the (unique up to additive constant) power series of $f(z)$ and prove that $f(z)$ can be analytically continued to $|z| < 1$.
   (b) (5 points) (10 points) Prove that $f(z)$ cannot be analytically continued to any connected open set that contains $|z| < 1$ and is strictly larger than it.
   (Hint: Consider boundary points of the form $z = e^{2\pi i \frac{a}{2^k}}$.)

6. (10 points) Let $f\colon \mathbb{D}(0, R) \to \mathbb{D}(0, R)$ be a holomorphic function.
   (a) (4 points) Prove that
$$\left| \frac{f(z) - f(0)}{R^2 - \overline{f(0)} f(z)} \right| \leq \frac{|z|}{R^2}$$
   (b) (3 points) Prove that if f has two fixed points then $f(z) = z$.
   (c) (3 points) Construct such a function f which does not have a fixed point.
   (Hint: Consider the upper half plane.)