UGA High School Varsity Math Tournament
October 26, 2024

WITH SOLUTIONS

Team Round

Time: 1 hour
Length: 3 problems

Max Score 210 points
70 points for a correct answer.

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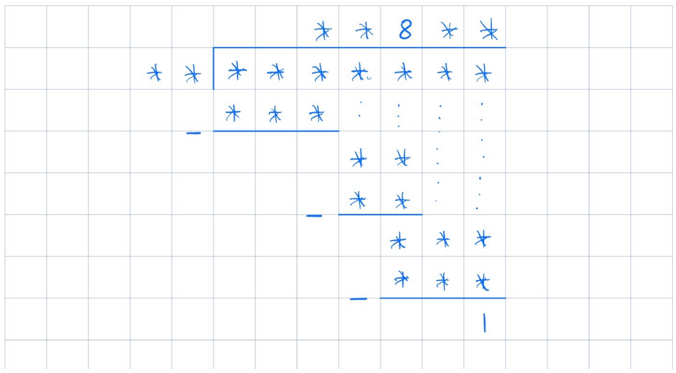
Problem 1. Alice found one of her mother's old notebooks. On one of the pages, she recognizes a long division but unfortunately only two of the digits are legible, the other ones look like little stars.

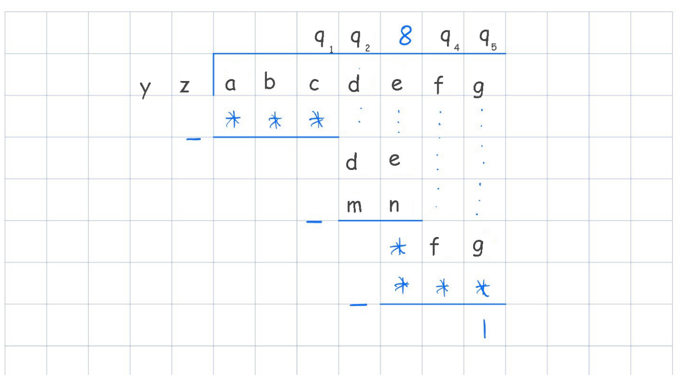


What is the dividend (the 7 digit number)?

Answer. 1089709

Solution. Some notation: we will call the two digit divisor yz, the five digit quotient $q_1q_2q_3q_4q_5$ and the seven digit dividend (the number we need to find) $abcdefg$.

We're given that $q_3 = 8$ so the long division looks like this:



Notice that we have also named one of the intermediate numbers mn, and have identified a few other $*$.

Some observations

- $yz > ab$
- $q_1 \cdot yz = abc$
- $q_2 = 0$

- $mn = 8 \cdot yz$

In particular, $8yz$ is a two digit number (i.e. $yz \leq 12$), while $q_1 yz$ is a three digit number. So $q_1 = 9$ and $yz = 12$. This means that $mn = 96$, and $de = 97$, 98 or 99.

More observations,

- $q_4 = 0$
- $q_5 \cdot 12$ is a three digit number, so $q_5 = 9$.

So we have $12 \cdot 90809 + 1 = abcdefg = 1089709$.

In summary, the completed long division is

$$
\begin{array}{r@{\,}r@{}c@{}c@{}c@{}c@{}c@{}c}
 & & & & 9 & 0 & 8 & 0 & 9 \\ \cline{3-9}
12 & \big) & 1 & 0 & 8 & 9 & 7 & 0 & 9 \\
 & & \underline{1} & \underline{0} & \underline{8} & & & & \\
 & & & & & 9 & 7 & & \\
 & & & & & \underline{9} & \underline{6} & & \\
 & & & & & & 1 & 0 & 9 \\
 & & & & & & \underline{1} & \underline{0} & \underline{8} \\
 & & & & & & & & 1
\end{array}
$$

Source: Losanges 39.

Problem 2. Consider the sets $\{x \bmod 37, 10x \bmod 37, 100x \bmod 37\}$ for integers $x = 1, \dots, 36$. What is the sum of the medians of these distinct sets?

Here, for an integer x, we write $x \bmod 37$ to be the unique integer a with $0 \le a < 37$ such that $x - a$ is divisible by 37.

Answer. 222

Solution.

Solution 1 Writing out all these sets is hardly difficult. For example, starting with $a = 1$, we have $10a = 10$ and $10 \bmod 37 = 10$, and $100a = 100$ and $100 \bmod 37 = 26$, giving us $\{1, 10, 26\}$ as one set. In a similar manner, we can work out all our distinct sets, and there are exactly 12 of them:

$$\begin{array}{lll} \{1, \mathbf{10}, 26\} & \{6, \mathbf{8}, 23\} & \{14, \mathbf{29}, 31\} \\ \{2, \mathbf{15}, 20\} & \{7, \mathbf{33}, 34\} & \{17, \mathbf{22}, 35\} \\ \{3, \mathbf{4}, 30\} & \{9, \mathbf{12}, 16\} & \{18, \mathbf{24}, 32\} \\ \{5, \mathbf{13}, 19\} & \{11, \mathbf{27}, 36\} & \{21, \mathbf{25}, 28\} \end{array}$$

Adding the medians then gives us

$$10 + 15 + 4 + 13 + 8 + 33 + 12 + 27 + 29 + 22 + 24 + 25 = 222.$$

Solution 2 We can do this a bit more cleverly.

The first observation we make is that all these sets are disjoint, and thus form a partition of the set $\{1, 2, \dots, 36\}$ into twelve subsets of 3. To see this, if we replace $x \bmod 37$ with $10x \bmod 37$, we get the set $\{10x \bmod 37, 100x \bmod 37, 1000x \bmod 37\}$. However, notice that 37 divides 999, so that means that $1000x \bmod 37 = x \bmod 37$ for all x. Repeating this argument with $10x$ in place of x tells us that the sets generated by $10x \bmod 37$ and $100x \bmod 37$ are the same as the set generated by $x \bmod 37$.

Now observe that $10(37 - x) \bmod 37 = -10x \bmod 37$ and $100(37 - x) \bmod 37 = -100x \bmod 37$. Now, for any integer y, if $y \bmod 37 \neq 0$, then $-y \bmod 37 = 37 - (y \bmod 37)$. Since $10x \bmod 37 \neq 0$ and $100x \bmod 37 \neq 0$ for $x = 1, 2, \dots, 36$ we deduce that if one of the sets is $\{a, b, c\}$, then $\{37 - a, 37 - b, 37 - c\}$ is another set. Moreover, if $a < b < c$ so that b is the median, then $37 - a > 37 - b > 37 - c$ and so $37 - b$ is the median of the corresponding other set.

Thus, we can pair up the 12 sets into six pairs $\{a, b, c\}, \{37 - a, 37 - b, 37 - c\}$ whose medians add up to 37; the total of all medians is then $6 \cdot 37 = 222$.

Remark Some background/motivation for this question: what happens when you look at the multiples $27x$ for x in the given set, i.e., $27 \cdot 1, 27 \cdot 2, \dots, 27 \cdot 36$? For example, looking at the first set $\{1, 10, 26\}$, we have $27 \cdot 1 = 027, 27 \cdot 10 = 270, 27 \cdot 26 = 702$. These numbers are, if we allow for leading 0s, cyclic permutations of each other! The same phenomenon, as one can check, is true for each of the other sets. This is no coincidence.

The key observation is that $1/37 = 27/999 = 0.\overline{027}$ where the overlined digit string repeats infinitely. Then for $x = 1, 2, \dots, 36$, if $27x = ABC$ for decimal digits A, B, C, we have $x/37 = 27x/999 = 0.\overline{ABC}$.

Now, if we have $x/37 = 0.\overline{ABC}$ then we have $10x/37 = A.\overline{BCA} = A + 0.\overline{BCA}$. However, we can also write $10x = 37q + r$ where $0 \leq r < 37$ by the usual division algorithm, so $10x/37 = q + r/37$. This tells us that $q = A$, and moreover, $r = 10x \bmod 37$, so we have $r/37 = (10x \bmod 37)/37 = 0.\overline{BCA}$. This means that $27(10x \bmod 37) = BCA$, which explains the digit-cycling phenomenon observed earlier.

Problem 3. A baseball diamond consists of four bases, named home, first base, second base, and third base. Normally you run from home, to first, to second, to third, and back to home. Suppose we have two baseball diamonds, called A and B, which share a home plate. Suppose Alice starts on home plate, and her friend Bob keeps track of how many laps she makes around each baseball diamond. However, any laps she runs in the wrong direction Bob counts as negative laps.

Alice has a peculiar method for running the bases. From any base, she will randomly select an adjacent base to run to. The fourth time Alice returns to home plate, what is the probability that Bob's count is 0 laps around diamond A and 0 laps around diamond B?

Answer. $\frac{6057}{2^{14}}$

Solution. Suppose at first that Alice's first step in her random walk is to go to first base in diamond A. Let p be the probability that she completes a lap before returning to home plate. She has a $\frac{1}{2}$ chance of immediately returning home, and a $\frac{1}{2}$ chance of going to second base. Assuming she makes it to second base, by symmetry she then will have a $\frac{1}{2}$ chance of completing the lap, and a $\frac{1}{2}$ chance of returning to home without completing the lap. Thus, $p = \frac{1}{4}$.

Therefore, the first time Alice returns to home plate, there is a $\frac{3}{4}$ chance that Bob's count is $(0, 0)$, and a $\frac{1}{4}$ chance that the count is one of $(\pm 1, 0)$, $(0, \pm 1)$, each with equal probability.

Note that at this point, we are taking a random walk on the integer lattice where we have a $\frac{3}{4}$ chance of staying still, and a $\frac{1}{16}$ for each of the 4 directions we can move in. In order to return to $(0, 0)$ after 4 iterations, we must have one of the following cases:

1. Bob's count stays the same all four times Alice returns home.

2. Bob's count changes twice and stays the same twice, with the two changes cancelling out.

3. Bob's count changes all four times, with the second two changes cancelling out the first two.

We see that

$$\text{Prob(Case 1)} = \left(\frac{3}{4}\right)^4 = \frac{81}{256},$$

For Case 2, we may assume without loss of generality that the first two times Alice returns to home Bob's count remains at $(0, 0)$, and multiply the resulting probability by $\binom{4}{2} = 6$. The chance to remain at $(0, 0)$ for the first two iterations is $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$, and the chance of the count then changing for the third iteration, and cancelling the change with the fourth iteration, is $\frac{1}{4} \cdot \frac{1}{16} = \frac{1}{64}$. Thus,

$$\text{Prob(Case 2)} = 6 \cdot \frac{9}{16} \cdot \frac{1}{64} = \frac{27}{512}$$

Finally, for case 3, note that the probability of the count changing all four times Alice returns home is $\left(\frac{1}{4}\right)^4 = \frac{1}{256}$. Assuming this happens, lets consider the probability of each possible outcome of the first two iterations. The possibilities for Bob's count are

$$\text{Prob}(0,0) = \frac{1}{4},$$

$$\text{Prob}(2,0) = \text{Prob}(-2,0) = \text{Prob}(0,2) = \text{Prob}(0,-2) = \frac{1}{16}$$

$$\text{Prob}(1,1) = \text{Prob}(1,-1) = \text{Prob}(-1,1) = \text{Prob}(-1,-1) = \frac{1}{8}$$

Thus, the probability that the second two moves cancel out the first two is

$$\frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{16} + \frac{1}{2} \cdot \frac{1}{8} = \frac{9}{64}.$$

Thus,

$$\text{Prob(Case 3)} = \frac{1}{256} \cdot \frac{9}{64} = 9 \cdot 2^{-14}$$

Finally, we see that the probability we originally wanted, the probability that Bob's count is $(0,0)$ the fourth time Alice returns to home plate, is

$$3^4 \cdot 2^{-8} + 3^3 \cdot 2^{-9} + 3^2 \cdot 2^{-14} = 3^2 \cdot 2^{-14} \cdot (9 \cdot 64 + 3 \cdot 32 + 1)$$

$$= 3^2 \cdot 2^{-14} \cdot 673$$

Authors. Paco Adajar, Rishika Agrawal, Jimmy Dillies, Mo Hendon, Gerasim Iliev, Tekin Karadag, Brian McDonald, David Plaxco, Paul Pollack, and Akash Singha Roy