UGA High School Varsity Math Tournament
October 26, 2024

WRITTEN TEST

TIME: 90 MINUTES
LENGTH: 25 PROBLEMS

MAX SCORE 250 POINTS
10 POINTS FOR A CORRECT ANSWER,
0 POINTS FOR AN INCORRECT ANSWER,
AND 2 POINTS FOR AN ANSWER LEFT BLANK.

No calculators, slide rules, or any other such instruments are allowed.
Scratchwork may be done on the test and on the three blank pages at the end.
Credit will be given only for answers marked on the scantron sheet.
If you finish the exam before time is called, turn in your scantron sheet to the person in the front and then exit quietly.
If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.
You will receive 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

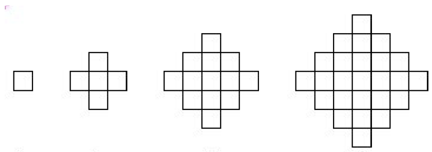
Problem 1. What is the last digit of $3^{2024} - 2^{2024}$?

(A) 0 (B) 1 (C) 3 (D) 5 (E) 9

Problem 2. How many subsets of $A = \{1, 2, 3, \ldots, 9, 10\}$ contain at least 3 of the elements of $B = \{1, 2, 3, 4, 5\}$?

(A) 32 (B) 320 (C) 480 (D) 512 (E) 1024

Problem 3. These 4 figures below contain respectively of 1, 5, 13, and 25 unit squares. If the process were to continue in the same manner, how many unit squares would there be in the 99th figure?"



(A) 10401 (B) 19405 (C) 20201 (D) 39805 (E) 40801

Problem 4. Given any real number x, we are sure that $|\sin|x||$ equals...

(A) $\sin x$ (B) $|\sin x|$ (C) $\sin|x|$ (D) $-\sin|-x|$ (E) $\cos|\frac{\pi}{2} - x|$

Problem 5. For how many values of n, where n is a positive integer, is the fraction $\frac{21n+4}{14n+3}$ non reduced.

(A) 0 (B) 1 (C) 2 (D) 7 (E) Infinitely many

Problem 6. The rational number 7.3 can be written as

$$x + \cfrac{1}{y + \frac{1}{z}}$$

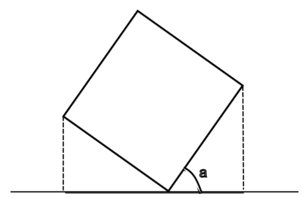
where x, y and z are positive integers. What is z?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 8

Problem 7. UGA's Calculus I classes have a maximum of 25 students. In one of the classes, 31% of the students come from out of state and 94% want to study engineering. Given that these percentages have been rounded to the nearest integer, how many students are in that class?

(A) 16 (B) 17 (C) 18 (D) 19 (E) 20

Problem 8. An square of side one is rotated by an angle a. Its orthogonal projection onto the x axis has length $\sqrt{\frac{3}{2}}$. What is the minimal value of a?



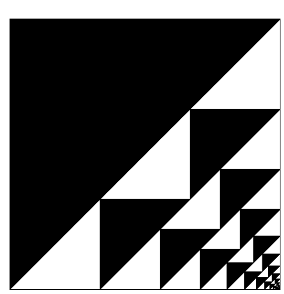
(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{8}$ (D) $\frac{\pi}{9}$ (E) $\frac{\pi}{12}$

Problem 9. Let $d(n)$ be the number of divisors of n, i.e. the number of positive integers that divide n. For example, $d(5) = 2$ and $d(6) = 4$. Compute the sum of the digits in E where

$$E = \sum_{n=1}^{2024} (-1)^{d(n)}.$$

(A) 11 (B) 13 (C) 15 (D) 17 (E) 19

Problem 10. What is the colored proportion of this square?



(A) $\frac{3}{4}$ (B) $\frac{3}{5}$ (C) $\frac{5}{6}$ (D) $\frac{7}{10}$ (E) $\frac{9}{10}$

Problem 11. The equation

$$x + \sqrt{x-1} + \sqrt{x+1} + \sqrt{x^2-1} = 4$$

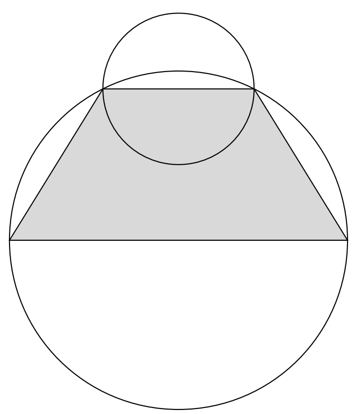
has a rational solution which in reduced form can be written as $\frac{a}{b}$. What is $a+b$?

(A) 7 (B) 8 (C) 9 (D) 11 (E) 16

Problem 12. An ideal Georgia license plate consists of any 3 letters followed by exactly 4 digits. To the three digits on the left, we can associate the number obtained by concatenating the positions of the letters in the alphabet. E.g. ABZ yields the number 1226, UGA yields 2171 and AAA gives 111. How many license plates are such that the 4 right digits correspond to the three initial letters? Beware, although AAA 0111 is a valid GA license plate, it does not meet the requirements of this problem, since we do not allow leading 0s.

(A) 1377 (B) 4131 (C) 4800 (D) 8232 (E) 9600

Problem 13. The smaller circle has diameter 1; this is also the distance between the centers of the two circles. The diameters of the two circles shown in the picture are parallel. What is the area of the shaded trapezoid?



(A) 2 (B) $\dfrac{1+\sqrt{3}}{2}$ (C) $\dfrac{1+\sqrt{5}}{2}$ (D) $\sqrt{3}$ (E) $\sqrt{5}$

Problem 14. Consider a regular decagon (i.e. a polygon with 10 sides). How many diagonals (i.e. segments joining non adjacent vertices) do not pass through its center?

(A) 30 (B) 35 (C) 40 (D) 45 (E) 60

Problem 15. Consider the polynomial $p(x) = x^2 + bx + c$ where b and c are integers. If $p(x)$ divides both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, what possible value can $p(1)$ have?

(A) -4 (B) -2 (C) 2 (D) 4 (E) 8

Problem 16. The Fibonacci sequence is built recursively by letting $F_0 = 0$, $F_1 = 1$ and defining each subsequent term $F_n = F_{n-1} + F_{n-2}$. For example, $F_2 = F_1 + F_0 = 1 + 0 = 1$ and $F_3 = F_2 + F_1 = 1 + 1 = 2$, etc.

Simplify

$$\tan^{-1}\left(\frac{F_{2023}}{F_{2024}}\right) + \tan^{-1}\left(\frac{F_{2022}}{F_{2025}}\right).$$

(A) $\frac{\pi}{12}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{3}$ (E) $\frac{\pi}{2}$

Problem 17. Alice has a machine labelled "invert", which performs an operation on binary strings, i.e. finite sequences of the digits 0 and 1. The invert machine switches every 0 to a 1 and vice versa. For example, if you feed the input "01111001 into the invert machine, you get "10000110". Alice starts with the string "0", and feeds it into the invert machine, which outputs "1". She then combines the input and output into the string "01", and feeds this as an input into the invert machine again, receiving output "10". She repeats this process, so that the first several numbers she feeds into the machine are

$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110$$

Let S be the 20th string she feeds into the machine, which then has 2^{19} digits. Bob comes in and looks at S and writes down 6 consecutive digits, starting with the 2024th digit. He then interprets the 6 digits as an integer written in binary, and converts it to base 10. What is Bob's integer?

(A) 18 (B) 12 (C) 44 (D) 50 (E) 56

Problem 18. Let $\log x$ denote the natural logarithm of x and let

$$A = (\log x)^{\log\log\log x}$$

$$B = x$$

$$C = (\log\log x)^{\log\log x}$$

$$D = (\log\log\log x)^{\log x}$$

$$E = \log x$$

Which of the following chains of inequalities is true for all large enough values of x?

(A) $E \leq A \leq C \leq D \leq B$

(B) $E \leq B \leq C \leq A \leq D$

(C) $E \leq A \leq C \leq B \leq D$

(D) $E \leq B \leq A \leq C \leq D$

(E) $E \leq B \leq D \leq C \leq A$

Problem 19. Consider a triangle whose sides are 5, 12 and 13. What is the sum of diameters of the inscribed and circumscribed circles?

(A) 16 (B) 17 (C) 18 (D) 18.5 (E) 19

Problem 20. Given that $1 + x + x^2 = 0$, what is the last digit of

$$x^{3036} + x^{2024} + x^{1012}?$$

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

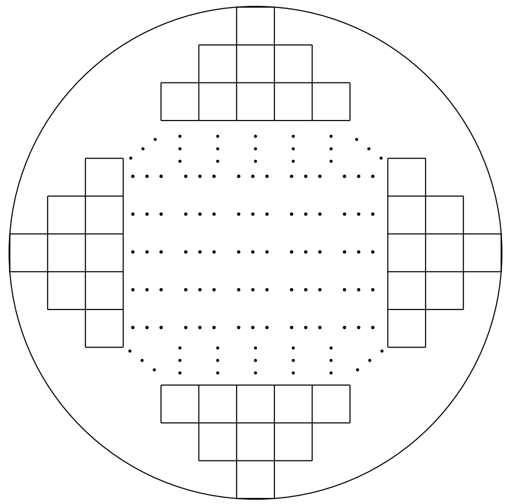
Problem 21. Let N be the product of all odd numbers between 1 and $100,000$. What is the sum of the last two digits of N?

(A) 6 (B) 7 (C) 11 (D) 12 (E) 17

Problem 22. What are the final (rightmost) two decimal digits in $2^{3^{5^{7^{\cdot^{\cdot^{\cdot^{97}}}}}}}$? Here the tower of exponents involves all of the primes up to 100, in increasing order.

(A) 08 (B) 68 (C) 24 (D) 76 (E) 28

Problem 23. Congruent squares are packed into a circle of radius 1 in a diamond arrangement with 2025 rows, as follows:



(Two vertices of the topmost, bottom-most, leftmost and rightmost squares lie on the circle.)

What is the total area of the squares?

(A) 1 (B) 2 (C) $\frac{2025}{1013}$ (D) $\frac{3961}{1936}$ (E) $\frac{2050313}{1024144}$

Problem 24. The *Collatz function* f maps even numbers n to $n/2$ and odd numbers n to $3n + 1$. Starting from an arbitrary positive integer n, the *up-down-pattern* associated to n is an infinite sequence of letters U and D defined as follows. If $n < f(n)$, the leftmost letter is a U; otherwise, it is a D. If $f(n) < f(f(n))$, the second letter is a U; otherwise, it is a D... and so on. For how many positive integers $n \leq 2024$ does the up-down-pattern of n begin DUDDU ?

(A) 63 (B) 64 (C) 126 (D) 127 (E) 506

Problem 25. Let PQR be a right triangle with legs PQ and PR of lengths 3 and 2 respectively, and with the right angle at vertex P. Square $PABC$ is constructed with vertices A, B, C lying on sides PQ, QR, RP respectively. What is the radius of the incircle of ΔABQ?

(A) $r = \frac{3}{13}(5 - \sqrt{10})$ (B) $r = \frac{3}{13}(10 - \sqrt{5})$ (C) $r = \frac{3}{10}(5 - \sqrt{13})$

(D) $r = \frac{3}{5}(10 - \sqrt{13})$ (E) $r = \frac{3}{5}(13 - \sqrt{10})$