UGA High School Varsity Math Tournament
October 26, 2024

WITH SOLUTIONS

WRITTEN TEST

TIME: 90 MINUTES
LENGTH: 25 PROBLEMS

MAX SCORE 250 POINTS
10 POINTS FOR A CORRECT ANSWER,
0 POINTS FOR AN INCORRECT ANSWER,
AND 2 POINTS FOR AN ANSWER LEFT BLANK.

Problem 1. What is the last digit of $3^{2024} - 2^{2024}$?

(A) 0 (B) 1 (C) 3 (D) 5 ♡ (E) 9

Solution. Notice that modulo 10, the successive powers of 2 are $2, 4, 8, 6, 2, \ldots$ and the successive powers of 3 are $3, 9, 7, 1, 3, \ldots$. In each case, the pattern is periodic with a repetition every 4. As 2024 is a multiple of 4, the answer is the same as for $3^4 - 2^4$ i.e. $1 - 6 = 5$ modulo 10.

Problem 2. How many subsets of $A = \{1, 2, 3, \ldots, 9, 10\}$ contain at least 3 of the elements of $B = \{1, 2, 3, 4, 5\}$?

(A) 32 (B) 320 (C) 480 (D) 512 ♡ (E) 1024

Solution. A first approach is brute force:

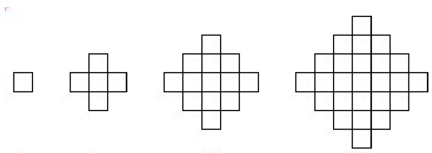
1. The number of subsets containing exactly 3 elements of B: $\binom{5}{3}2^5 = 10 \cdot 2^5$
2. The number of subsets containing exactly 4 elements of B: $\binom{5}{4}2^5 = 5 \cdot 2^5$
3. The number of subsets containing exactly 5 elements of B: $\binom{5}{5}2^5 = 1 \cdot 2^5$

So the total is $(10 + 5 + 1)2^5 = 2^9 = 512$.

Note that this is exactly half of the total number of subsets of A. This suggests a simpler solution. Here's one:

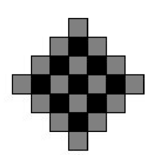
Pick a subset of A that contains at least 3 elements of B. Its complement is also a subset of A but which contains 2 or less elements of B. This gives a bijection between the subsets we want to count and all other subsets, i.e. exactly half of the subsets have at least 3 elements of B.

Problem 3. These 4 figures below contain respectively of 1, 5, 13, and 25 unit squares. If the process were to continue in the same manner, how many unit squares would there be in the 99th figure?"



(A) 10401 (B) 19405 (C) 20201 (D) 39805 (E) 40801

Solution. **1.** Figure n consists of a square of side $2n-1$ from which we remove four corners with $1+2+\cdots+(n-1)$ squares. This yields a total of $(2n-1)^2 - 4\frac{n(n-1)}{2} = 2n^2 - 2n + 1$ squares in figure n which for 99 is 19405.
2. If we tilt the figure sideways, we see that it is made of two intertwined 'layers' of squares, each one made of smaller squares touching at the corners:



The largest one has side n, the smallest one $n-1$, all in all that's $n^2 + (n-1)^2 = 2n^2 - 2n + 1$, etc. Source: OMB

Problem 4. Given any real number x, we are sure that $|\sin|x||$ equals...

(A) $\sin x$ (B) $|\sin x|$ ♡ (C) $\sin|x|$ (D) $-\sin|-x|$ (E) $\cos|\frac{\pi}{2}-x|$

Solution. Since $\sin(-x) = -\sin(x)$, $|\sin|x|| = |\sin x|$.

Problem 5. For how many values of n, where n is a positive integer, is the fraction $\frac{21n+4}{14n+3}$ non reduced.

(A) 0 ♡ (B) 1 (C) 2 (D) 7 (E) Infinitely many

Solution. Recall that two numbers a and b are relatively prime if and only if a and $b-a$ are relatively prime; this is the essence of the Euclidean algorithm. If we start with $21n+4$ and $14n+3$, we obtain successively the pairs $14n+3$ and $7n+1$, $7n+2$ and $7n+1$. The last two are clearly mutually prime hence the initial fraction is always reduced.

Problem 6. The rational number 7.3 can be written as

$$x + \frac{1}{y+\frac{1}{z}}$$

where x, y and z are positive integers. What is z?

(A) 1 (B) 2 (C) 3 ♡ (D) 4 (E) 8

Solution. Since $\frac{1}{y+\frac{1}{z}} < 1$, $x = 7$. Hence, $0.3 = \frac{1}{y+\frac{1}{z}}$ or, taking the reciprocal, $3 + \frac{1}{3} = y + \frac{1}{z}$ therefore $z = 3$.

Problem 7. UGA's Calculus I classes have a maximum of 25 students. In one of the classes, 31% of the students come from out of state and 94% want to study engineering. Given that these percentages have been rounded to the nearest integer, how many students are in that class?

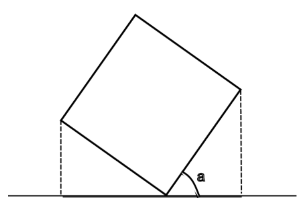
(A) 16 (B) 17 (C) 18 (D) 19 (E) 20

Solution. Let n be the total number of students. As $1/25 = 0.04$ each students contributes at least 4% and 2 students contribute at least 8% this means that the 6% of students who are not studying engineering correspond to a single student. More precisely, The 6% of students who do not want to study engineering, tell us that

$$\frac{6.5}{100} \geq \frac{1}{n} > \frac{5.5}{100}.$$

Hence $15.3... < 16 \leq n \leq 18 < 18.1....$ As each student contributes at least 5.5%, 6 students amount to a 33% contribution which is too high. Conversely, 4 students contribute at most 6.5% each, i.e. 26%, which is too small. Thus the 34% of out state students represent exactly 5 persons. As $\frac{5}{17}$ and $\frac{5}{18}$ are clearly too small, we are left with $\frac{5}{16}$ which is precisely 31.25%. Thus the number of students is equal to 16. Source: Inspired by Dutch Math Olympiad 2024

Problem 8. An square of side one is rotated by an angle a. Its orthogonal projection onto the x axis has length $\sqrt{\frac{3}{2}}$. What is the minimal value of a?



(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{8}$ (D) $\frac{\pi}{9}$ (E) $\frac{\pi}{12}$

Answer. $\frac{\pi}{12}$.

Solution. The projection onto the x axis has length $\cos a + \sin a$. The latter expression can be rewritten as the sine of an angle:

$$\cos a + \sin a = \sqrt{2}\left(\frac{\sqrt{2}}{2}\cos a + \frac{\sqrt{2}}{2}\sin a\right) = \sqrt{2}\sin\left(a + \frac{\pi}{4}\right)$$

whence $\sin\left(a + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$. This means that $a + \frac{\pi}{4} = \frac{\pi}{3}$ and $a = \frac{\pi}{12}$.

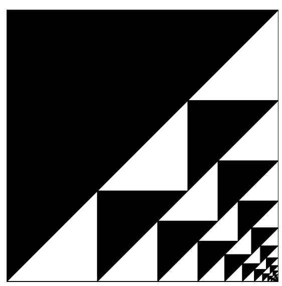
Problem 9. Let $d(n)$ be the number of divisors of n, i.e. the number of positive integers that divide n. For example, $d(5) = 2$ and $d(6) = 4$. Compute the sum of the digits in E where

$$E = \sum_{n=1}^{2024} (-1)^{d(n)}.$$

(A) 11 (B) 13 (C) 15 (D) 17 (E) 19

Solution. Given an integer, n, its factors come in pairs : a and $\frac{n}{a}$. Hence if n is a perfect square if and only if it has an odd number of divisors; in other words, if $d(n)$ is odd. There are 44 perfect squares between 1 and 2024 therefore $\sum_{n=1}^{2024}(-1)^{d(n)} = (-1) \times 44 + 1. \times (2024 - 44) = 1936$ whose digits add up to 19.

Problem 10. What is the colored proportion of this square?



(A) $\frac{3}{4}$ (B) $\frac{3}{5}$ (C) $\frac{5}{6}$ (D) $\frac{7}{10}$ (E) $\frac{9}{10}$

Solution. Clearly the upper left half of the square is colored. Each diagonal strip underneath is cut into 5 triangles of equal area, 2 of which are colored. In other words, $\frac{2}{5}$ of the lower half are colored. In total, this is $\frac{1}{2} + \frac{1}{2} \cdot \frac{2}{5} = \frac{7}{10}$

Problem 11. The equation

$$x + \sqrt{x-1} + \sqrt{x+1} + \sqrt{x^2-1} = 4$$

has a rational solution which in reduced form can be written as $\frac{a}{b}$. What is $a + b$?

(A) 7 (B) 8 (C) 9 (D) 11 (E) 16

Answer. The solution to this equation is $\frac{5}{4}$ hence $a + b = 9$.

Note that

$$(\sqrt{x-1}+\sqrt{x+1})^2 = 2x + 2\sqrt{x^2-1}$$

so if we let $u = \sqrt{x-1}+\sqrt{x+1}$, the above equation reduces to

$$u + \frac{u^2}{2} = 4$$

whose solutions are $u = -4$ and $u = 2$. As u represents a positive quantity, only the latter solution is relevant, i.e.

$$\sqrt{x-1}+\sqrt{x+1} = 2.$$

This implies that

$$2x + 2\sqrt{x^2-1} = 4$$

or

$$2 - x = \sqrt{x^2-1}.$$

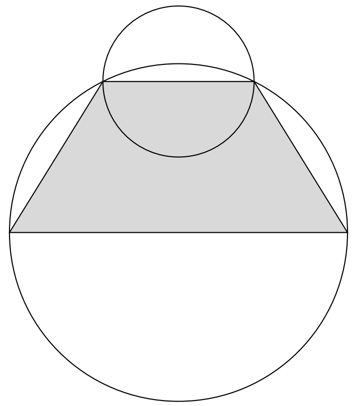
Squaring this expression yields the linear equation $4 - 4x = -1$, i.e. $x = \frac{5}{4}$.

Problem 12. An ideal Georgia license plate consists of any 3 letters followed by exactly 4 digits. To the three digits on the left, we can associate the number obtained by concatenating the positions of the letters in the alphabet. E.g. ABZ yields the number 1226, UGA yields 2171 and AAA gives 111. How many license plates are such that the 4 right digits correspond to the three initial letters? Beware, although AAA 0111 is a valid GA license plate, it does not meet the requirements of this problem, since we do not allow leading 0s.

(A) 1377 (B) 4131 ♡ (C) 4800 (D) 8232 (E) 9600

Solution. Note that the letters on the left define uniquely the four digits on the right, so all we need to do is count the valid number of combinations of letters on the left. Since there are four digits, exactly one of the letters should be in position 10 to 26 and two should be in position 1 to 9. There are $9 \times 9 \times 17 \times 3 = 4131$ different license plates. The 3 comes from the possible permutations of the 3rd letter.

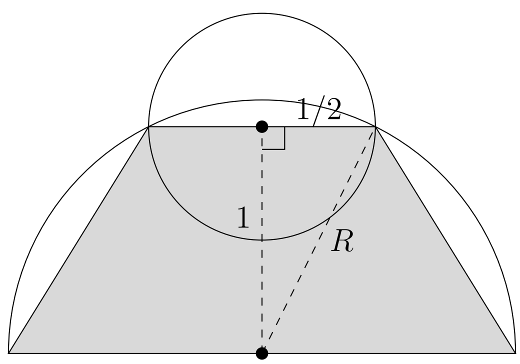
Problem 13. The smaller circle has diameter 1; this is also the distance between the centers of the two circles. The diameters of the two circles shown in the picture are parallel. What is the area of the shaded trapezoid?



Ⓐ 2 Ⓑ $\dfrac{1+\sqrt{3}}{2}$ Ⓒ $\dfrac{1+\sqrt{5}}{2}$ Ⓓ $\sqrt{3}$ Ⓔ $\sqrt{5}$

Answer. $\dfrac{1+\sqrt{5}}{2}$

Solution. Let R be the radius of the larger circle. Joining the centers of the two circles with a line segment and drawing a radius from the center of the larger circle to an endpoint of the diameter of the smaller circle, we have a right triangle, as shown below, with legs 1/2 and 1.



By the Pythagorean theorem, we then get that $(1/2)^2 + 1^2 = R^2$, so $R = \sqrt{5}/2$. The larger triangle thus has diameter $\sqrt{5}$. The trapezoid then has parallel sides 1 and $\sqrt{5}$, and height 1, so its area is $\frac{1}{2} \cdot (1+\sqrt{5}) = \frac{1+\sqrt{5}}{2}$.

Problem 14. Consider a regular decagon (i.e. a polygon with 10 sides). How many diagonals (i.e. segments joining non adjacent vertices) do not pass through its center?

Ⓐ 30 Ⓑ 35 Ⓒ 40 Ⓓ 45 Ⓔ 60

Solution.

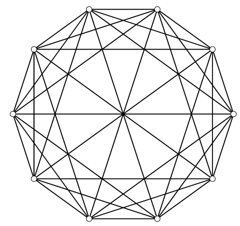


Figure 1: Some of the diagonals and some that do not count...

There are $\binom{10}{2} = 45$ segments joining two vertices. Of those, 5 pass through the center and 10 are actual edges so the final count is $45 - 5 - 10 = 30$.

Problem 15. Consider the polynomial $p(x) = x^2 + bx + c$ where b and c are integers. If $p(x)$ divides both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, what possible value can $p(1)$ have?

(A) -4 (B) -2 (C) 2 (D) 4 ♡ (E) 8

Solution. Since $p(x)$ divides both polynomials, it also divides any linear combination of them. In particular, $p(x)$ divides $3(x^4 + 6x^2 + 25) - (3x^4 + 4x^2 + 28x + 5) = 14x^2 - 28x + 70 = 14(x^2 - 2x + 5)$. Since the leading coefficient of $p(x) = 1$, $p(x)$ actually equals $x^2 - 2x + 5$ and $p(1) = 4$.

Problem 16. The Fibonacci sequence is built recursively by letting $F_0 = 0$, $F_1 = 1$ and defining each subsequent term $F_n = F_{n-1} + F_{n-2}$. For example, $F_2 = F_1 + F_0 = 1 + 0 = 1$ and $F_3 = F_2 + F_1 = 1 + 1 = 2$, etc.

Simplify

$$\tan^{-1}\left(\frac{F_{2023}}{F_{2024}}\right) + \tan^{-1}\left(\frac{F_{2022}}{F_{2025}}\right).$$

(A) $\frac{\pi}{12}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{4}$ ♡ (D) $\frac{\pi}{3}$ (E) $\frac{\pi}{2}$

Solution. Taking the tangent of the above expression

$$\tan\left(\tan^{-1}\left(\frac{F_{2023}}{F_{2024}}\right) + \tan^{-1}\left(\frac{F_{2022}}{F_{2025}}\right)\right)$$

and using Simpson's formula for the tangent of a sum we can reduce it to

$$\frac{\frac{F_{2023}}{F_{2024}} + \frac{F_{2022}}{F_{2025}}}{1 - \frac{F_{2023}}{F_{2024}}\frac{F_{2022}}{F_{2025}}}$$

After clearing the denominators of the the numerator and the denominator of this fraction, we are left with

$$\frac{F_{2023}F_{2025} + F_{2022}F_{2024}}{F_{2024}F_{2025} - F_{2023}F_{2022}}.$$

Using the recursive formula, the numerator can be rewritten as

$$(F_{2024} - F_{2022})F_{2025} + F_{2022}(F_{2025} - F_{2023})$$

or

$$F_{2024}F_{2025} - F_{2023}F_{2022}.$$

This means that the tangent of the initial angle is 1, i.e. the expression equals $\frac{\pi}{4}$.

Note that this problem would have yielded the same answer using any four successive terms in the Fibonacci sequence. The alert reader could have taken the shorcut to compute the equivalent expression built with the first four terms in the Fibonacci sequence:

$$\tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}\left(\frac{0}{2}\right).$$

Problem 17. Alice has a machine labelled "invert", which performs an operation on binary strings, i.e. finite sequences of the digits 0 and 1. The invert machine switches every 0 to a 1 and vice versa. For example, if you feed the input "01111001 into the invert machine, you get "10000110". Alice starts with the string "0", and feeds it into the invert machine, which outputs "1". She then combines the input and output into the string "01", and feeds this as an input into the invert machine again, receiving output "10". She repeats this process, so that the first several numbers she feeds into the machine are

$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110$$

Let S be the 20th string she feeds into the machine, which then has 2^{19} digits. Bob comes in and looks at S and writes down 6 consecutive digits, starting with the 2024th digit. He then interprets the 6 digits as an integer written in binary, and converts it to base 10. What is Bob's integer?

(A) 18 (B) 12 (C) 44 (D) 50 (E) 56

Solution. There are a few reasonable approaches. First, we can observe (and prove by induction) that the nth digit of S is 1 precisely when the binary representation of $n-1$ has an odd number of 1's. To find the 2024th digit, we then express 2023 in binary,

$$2023 = 1024 + 512 + 256 + 128 + 64 + 32 + 4 + 2 + 1 = (11111100111)_2,$$

which has an odd number of 1's, so the 2024th digit in S is 1. We can then just count in binary to reveal the 6 digits in sequence,

11111100111, 11111101000, 11111101001, 11111101010, 11111101011, 11111101100,

which have, respectively, an odd, odd, even, even, odd, and even number of 1's, so the 6 digits recorded by Bob form the string "110010", which we can convert from binary to decimal to obtain $32 + 16 + 2 = 50$.

Now for a second approach; one can solve this problem without finding the rule for the n-th digit directly. We don't need to think about the 20th string Alice feeds into the machine, it is enough to consider the 12th string she feeds in, which has $2^{11} = 2048$

digits. As Alice continues the process, the first 2048 digits never change. Note that for the 12th string Alice feeds into the machine, reading the string backwards inverts the string the same way Alice's machine does. Thus, since $2024 = 2^{11} - 24$, we will be done if we can write down the first 25 digits of the string.

This is easy enough, we get for the first 32 digits:

$$0110100110010110100(101100)1101001$$

The ones in the parentheses are the ones that correspond to digits $2024 - 2029$ when read backwards and inverted, so again we find that the digits Bob writes down are 110010, and so Bob's number is 50.

Problem 18. Let $\log x$ denote the natural logarithm of x and let

$$A = (\log x)^{\log\log\log x}$$

$$B = x$$

$$C = (\log\log x)^{\log\log x}$$

$$D = (\log\log\log x)^{\log x}$$

$$E = \log x$$

Which of the following chains of inequalities is true for all large enough values of x?

(A) $E \le A \le C \le D \le B$

(B) $E \le B \le C \le A \le D$

(C) $E \le A \le C \le B \le D$

(D) $E \le B \le A \le C \le D$

(E) $E \le B \le D \le C \le A$

Solution. We rewrite each expression as the exponential of some expression involving iterated logarithms, using the rule $U^V = \exp(V \log U)$:

$$A = \exp((\log\log\log x)(\log\log x)),$$

$$B = \exp(\log x),$$

$$C = \exp((\log\log x)(\log\log\log x)),$$

$$D = \exp((\log x)(\log\log\log\log x)),$$

$$E = \exp(\log\log x).$$

We can order A, B, C, D, and E by ordering the arguments inside the exponential, since exp is a strictly increasing function.

First, observe that

$$\log\log x < (\log\log x)(\log\log\log x)$$

for all large x, showing that $E < A = C$ once x is large enough. Continuing, for all large enough x, we have $\log\log x < (\log x)^{1/2}$, and

$$(\log\log x)(\log\log\log x) < (\log\log x)^2 < ((\log x)^{1/2})^2 = \log x.$$

Thus, $C < B$ for large x. Finally,

$$\log x < (\log x)(\log\log\log\log x)$$

for large values of x, yielding $B < D$. Thus, once x is sufficiently large,

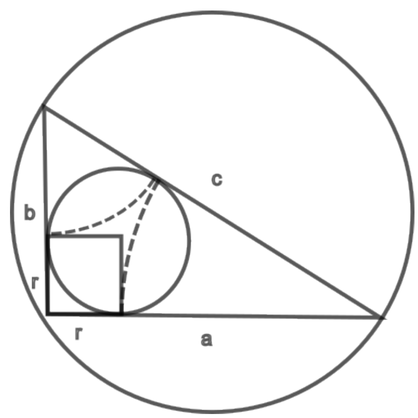
$$E < A = C < B < D.$$

The only answer choice compatible with this ordering is (C).

Problem 19. Consider a triangle whose sides are 5, 12 and 13. What is the sum of diameters of the inscribed and circumscribed circles?

(A) 16 (B) 17♡ (C) 18 (D) 18.5 (E) 19

Solution. Note that $5^2 + 12^2 = 13^2$ hence we are dealing with a right triangle of hypotenuse 13. Let ABC be the triangle with side $CA = 5, CB = 12$ and $AB = 13$, so the right angle is at C. The diameter of the circumscribed circle is the hypotenuse, which is 13. Moreover, if the incircle (i.e. inscribed circle) touches the sides CB, BA and AC at points D, E and F respectively, then since the tangents from a point to a circle have the same length, we find that $AE = AF = 5 - r$ and $BE = BD = 12 - r$, with r being the radius of the incircle. Hence $(5 - r) + (12 - r) = 13$, showing that the diameter of the incircle is $2r = 4$. Hence the sum of the diameters is $13 + 4 = 17$.



Problem 20. Given that $1 + x + x^2 = 0$, what is the last digit of

$$x^{3036} + x^{2024} + x^{1012}?$$

(A) 0♡ (B) 2 (C) 4 (D) 6 (E) 8

Solution. Note that $x^3 - 1 = (x - 1)(x^2 + x + 1)$ hence $x^3 = 1$. Now, $x^{3036} + x^{2024} + x^{1012} = (x^3)^{1012} + (x^3)^{674}x^2 + (x^3)^{337}x = 1 + x + x^2 = 0$.

Problem 21. Let N be the product of all odd numbers between 1 and $100,000$. What is the sum of the last two digits of N?

(A) 6 (B) 7♡ (C) 11 (D) 12 (E) 17

Solution. By the Chinese Remainder Theorem, it is enough to determine N modulo 4 and modulo 25. Modulo 25, N equals 0 as it is, among others, a multiple of 5 and of 15. Half of the odd numbers are congruent to 1 modulo 4 and half are congruent to $3 = -1$, thus

$$N = (-1)^{25000} 1^{25000} = 1 \bmod 4.$$

Since N is a multiple of 25, it ends in 00, 25, 50 or 75. Of these, only 25 equals 1 modulo 4. So the last two digits are 25 and their sum is 7.

The Chinese Remainder Theorem states that if you have a system of simultaneous congruences:

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_k \pmod{n_k} \end{aligned}$$

where $n_1, n_2, \ldots, n_k$ are pairwise coprime (each pair of n_i and n_j has no common factors other than 1), then there exists a unique solution x modulo $N = n_1 \cdot n_2 \cdot \ldots \cdot n_k$.

In other words, there is a unique integer x between 0 and $N - 1$ that satisfies all of these congruences at the same time. The theorem is useful for simplifying complex problems into smaller, easier-to-solve parts by working with the remainders separately.

Problem 22. What are the final (rightmost) two decimal digits in $2^{3^{5^{7^{\cdot^{\cdot^{\cdot^{97}}}}}}}$? Here the tower of exponents involves all of the primes up to 100, in increasing order.

(A) 08 ♡ (B) 68 (C) 24 (D) 76 (E) 28

Solution. Let N be the (rather large) integer defined by this tower of exponents. We are asking for the remainder when N is divided by 100: that is, we want N modulo 100. By the Chinese Remainder Theorem, it suffices to determine N modulo 4 and N modulo 25.

Clearly, $4 \mid N$, so that $N \equiv 0 \pmod 4$. To determine N modulo 25, we bring in some number theory: With ϕ denoting Euler's phi-function, we have $\phi(25) = 20$. Hence, $2^{20} \equiv 1 \pmod{25}$, and 2^m mod 25 depends only on m modulo 20. These observations reduce the problem to determining

$$3^{5^{\cdot^{\cdot^{\cdot^{97}}}}} \bmod 20.$$

Since $3^4 = 81 \equiv 1 \pmod{20}$, it is enough to determine

$$5^{7^{\cdot^{\cdot^{\cdot^{97}}}}} \bmod 4.$$

Since $5 \equiv 1 \pmod 4$, the integer represented by this last tower is 1 mod 4.

Working backwards,

$$3^{5^{\cdot^{\cdot^{\cdot^{97}}}}} \equiv 3^1 \equiv 3 \pmod{20}$$

and

$$N \equiv 2^3 \equiv 8 \pmod{25}.$$

Combining the congruences $N \equiv 8 \pmod{25}$ and $N \equiv 0 \pmod 4$ leads to $N \equiv 8 \pmod{100}$. Hence, the final two digits in N are 08.

The Euler's phi function, denoted as $\phi(n)$, is a function that counts the number of positive integers less than or equal to n that are coprime to n (i.e., their greatest common divisor with n is 1).

For example:

$$\phi(6) = 2,$$

since the integers 1 and 5 are the only numbers less than or equal to 6 that are coprime to 6.

The function has a useful formula when n can be expressed as a product of prime powers:

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_k}\right),$$

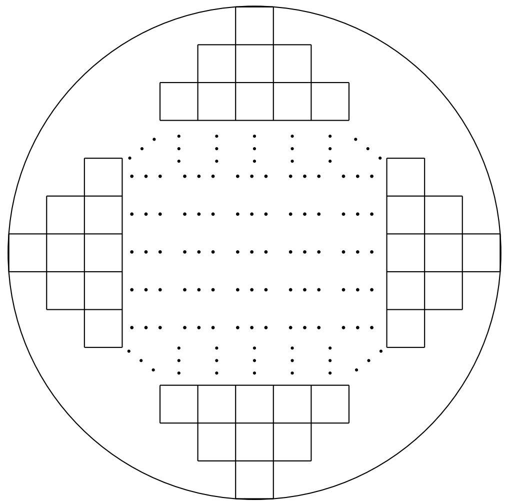
where $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ is the prime factorization of n.

For instance, if $n = 12$, the prime factorization is $2^2 \cdot 3$, so:

$$\phi(12) = 12\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 12 \cdot \frac{1}{2} \cdot \frac{2}{3} = 4.$$

The Euler's ϕ function is essential in number theory, particularly in the study of modular arithmetic.

Problem 23. Congruent squares are packed into a circle of radius 1 in a diamond arrangement with 2025 rows, as follows:



(Two vertices of the topmost, bottom-most, leftmost and rightmost squares lie on the circle.)

What is the total area of the squares?

(A) 1 (B) 2 (C) $\frac{2025}{1013}$ (D) $\frac{3961}{1936}$ (E) $\frac{2050313}{1024144}$

Solution. We show that in fact, the constant 2025 can be replaced by any odd positive integer. Let $2n - 1$ be the total number of rows/columns/squares in the central row/column of the formation.

Let s be the side length of one square. Note that the middle row of the formation is a $(2n-1)s \times s$ rectangle whose main diagonal is a diameter of the circle, and thus has length 2. By the Pythagorean theorem, we then have that

$$s^2 + [(2n-1)s]^2 = 2^2$$

and so $s^2 = 4/(4n^2 - 4n + 2) = 2/(2n^2 - 2n + 1)$.

Now, the total number of squares is exactly

$$1 + 3 + \cdots + (2n-3) + (2n-1) + (2n-3) + \cdots + 1 = n^2 + (n-1)^2 = 2n^2 - 2n + 1.$$

The total area of the squares is then $(2n^2 - 2n + 1)s^2 = 2$, independent of n.

Problem 24. The *Collatz function* f maps even numbers n to $n/2$ and odd numbers n to $3n + 1$. Starting from an arbitrary positive integer n, the *up-down-pattern* associated to n is an infinite sequence of letters U and D defined as follows. If $n < f(n)$, the leftmost letter is a U; otherwise, it is a D. If $f(n) < f(f(n))$, the second letter is a U; otherwise, it is a D. . . and so on. For how many positive integers $n \leq 2024$ does the up-down-pattern of n begin DUDDU ?

Ⓐ 63 Ⓑ 64 Ⓒ 126 Ⓓ 127♡ Ⓔ 506

Solution. Let $n_k = f^{(k)}(n)$. Since the last letter of DUDDU is U, we know that $n_4 < n_5$. Hence, the Collatz function sends n_4 to $n_5 = 3n_4 + 1$. This happens if and only if n_4 is odd, i.e., $n_4 \equiv 1 \pmod 2$.

Looking at the next-to-last letter, $n_3 > n_4$. So n_3 is sent by f to $n_4 = n_3/2$. In order for n_4 to be odd, it is necessary and sufficient that $n_3 = 2n_4$ belong to the residue class 2 mod 4.

Continuing to move left in the string, we see that $n_2 > n_3$. Thus, $n_2 = 2n_3$, which corresponds to requiring $n_2 \equiv 4 \pmod 8$.

As $n_1 < n_2$, we have $3n_1 + 1 = n_2 \equiv 4 \pmod 8$; solving gives $n_1 \equiv 1 \pmod 8$.

Finally, $n_0 > n_1$. Thus, $n = n_0 = 2n_1$, and this corresponds to the condition that $n \equiv 2 \pmod{16}$.

Thus, the question comes down to: How many positive integers not exceeding 2024 belong to the residue class 2 mod 16. This is the same as asking for the number of nonnegative integers j for which $2 + 16j \leq 2024$, i.e., $j \leq 126.375$. There are 127 of these.

Problem 25. Let PQR be a right triangle with legs PQ and PR of lengths 3 and 2 respectively, and with the right angle at vertex P. Square $PABC$ is constructed with vertices A, B, C lying on sides PQ, QR, RP respectively. What is the radius of the incircle of ΔABQ?

Ⓐ $r = \frac{3}{13}(5 - \sqrt{10})$ Ⓑ $r = \frac{3}{13}(10 - \sqrt{5})$ Ⓒ $r = \frac{3}{10}(5 - \sqrt{13})$♡
Ⓓ $r = \frac{3}{5}(10 - \sqrt{13})$ Ⓔ $r = \frac{3}{5}(13 - \sqrt{10})$

Solution. Letting a denote the length of a side of the square $PABC$, note that $AB = a$, $AQ = PQ - AP = 3 - a$, so that by the similarity of triangles AQB and PQR, we obtain

$$\frac{AQ}{PQ} = \frac{AB}{PR} \implies \frac{3-a}{3} = \frac{a}{2} \implies 1 - \frac{a}{3} = \frac{a}{2} \implies a = \frac{6}{5}.$$

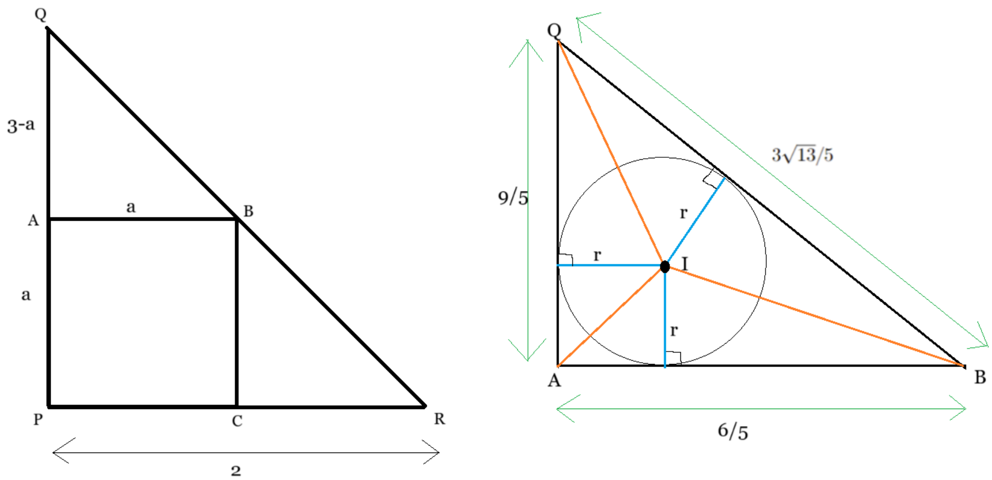
Thus $AB = a = 6/5$, $AQ = 3 - a = 9/5$ and by the Pythagorean theorem, $BQ = 3\sqrt{13}/5$.

Now let I be the incenter of ΔABQ. Join IA, IB, IQ and drop the perpendiculars from I onto the three sides of ΔABQ. Then the common length of each of these three perpendiculars is precisely the desired inradius r, so the areas of the three triangles IAQ, IAB and IBQ are $\frac{1}{2}r \cdot \frac{9}{5}$, $\frac{1}{2}r \cdot \frac{6}{5}$ and $\frac{1}{2}r \cdot \frac{3\sqrt{13}}{5}$ respectively. The sum of these three is the area of ΔABQ, which is $\frac{1}{2} \cdot \frac{6}{5} \cdot \frac{9}{5}$. Thus we must have

$$\frac{1}{2}r \cdot \frac{9}{5} + \frac{1}{2}r \cdot \frac{6}{5} + \frac{1}{2}r \cdot \frac{3\sqrt{13}}{5} = \frac{1}{2} \cdot \frac{6}{5} \cdot \frac{9}{5} \implies r = \frac{18}{5(5+\sqrt{13})} = \frac{3}{10}(5-\sqrt{13}).$$

Remark: An alternative way to compute the inradius of triangle ABQ after knowing its sides would be to proceed as in the solution to problem 19. Source:

Figure 2: Geometry Question 1



inspired from Euclid Contest 2007

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