UGA High School Varsity Math Tournament
October 26, 2024

WITH SOLUTIONS

Ciphering Round

Time: 2 minutes per problem
Length: 10 problems

Max Score 100 points
10 points for a correct answer.

Problem 1. Solve for a:

$$a^3 + 2^a = 2024.$$

Answer. 10

Solution. The function $a \mapsto a^3 + 2^a$ is increasing hence if there is a solution, it is unique. One checks quickly that $a = 10$ works.

Problem 2. What is the number of prime numbers between 100 and 1000 whose digits are in arithmetic progression? Recall that numbers are said to form an arithmetic progression if the difference between consecutive numbers is constant.

Answer. 0

Solution. Consider a three digit integer abc whose digits are in arithmetic progression, i.e. $b - a = c - b$. This implies that $a + c = 2b$ and therefore the sum of digits of that number is $c + b + a = 3b$, a multiple of 3 which means that the integer is multiple of 3 and never prime.

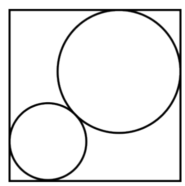
Problem 3. How many integer points lie on the closed line segment joining $(0,0)$ to $(20,24)$?

The point (x,y) is an integer point if both x and y are integers, e.g. $(0,0)$. A line segment is said to be closed when it includes both endpoints.

Answer. 5

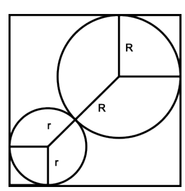
Solution. The line segment is defined by the equation $y = \frac{24}{20}x = \frac{6}{5}x$ and $0 \leq x \leq 20$. For (x,y) to be an integer, x must be an integer and and $\frac{6}{5}x$ must be an integer, i.e. x must be a multiple of 5. There are 5 such values for x: 0, 5, 10, 15 and 20.

Problem 4. Two circles are tangent internally at two points of a square of side one. The diameter of the larger circle is 11 times that of the smaller one. Given that the circles are mutually tangent, what is the sum of their radii? Write your answer in the form $a + b\sqrt{c}$ where a, b and c are integers.



Answer. $2 - \sqrt{2}$

Solution. We can add to the drawing the radii touching the points of tangency as well as the segment joining the centers of the circles:



If we call the radii r and R, we see that the projection onto one of the sides gives

$$1 = r + R + (r+R)\cos\frac{\pi}{4}.$$

This means that $1 = \left(1 + \frac{\sqrt{2}}{2}\right)(r+R)$ hence

$$r + R = \frac{1}{1 + \frac{\sqrt{2}}{2}} = 2 - \sqrt{2}$$

Note that the ratio 11 is irrelevant.

Problem 5. How many digits does the number $4^{16}5^{25}$ have (in base 10)?

Answer. 28

Solution. You can rewrite $4^{16}5^{25}$ as $2^{32}5^{25}$ or $2^7 10^{25}$. The former is 128 and has 3 digits to which we add 25 zeroes to get a total of 28 digits.

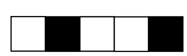
Problem 6. A multiset is a set where elements can be repeated, e.g. $\{2, 2, 5, 13\}$ is a multiset with 4 elements. As with sets, the order of the elements in a multiset does not matter.
Consider the three element multisets $\{a, b, c\}$ where $abc = 2024$ and a, b and c are positive integers. How many such multisets are there?

Answer. 16

Solution. The prime factorization of 2024 is $2^3.11.23$ hence there is no factorization with three identical numbers (i.e. 2024 is not a perfect cube) and there are two set with two identical factors, $\{2, 2, 506\}$ and $\{1, 1, 2024\}$; all other such sets have distinct elements. Let's assume for a moment that the order of the elements matters. We then have

$$\binom{5}{2} \cdot 3 \cdot 3 = 90$$

ways of breaking up the the above primes: 3 positions for 11 and 23 and $\binom{3+2}{2}$ ways of placing the twos:



The 2 black boxes determine the locations of the 2s: 1 for a, 2 for b and 0 for c.

Removing the 6 combinations with duplicates metionned above and dividing by the 3!, the number of possible rearrangements, we get a total of 14 factorizations with unique terms (up to a rearrangement). All in all we get

$$14 + 2 = 16$$

such sets.

Problem 7. There is a secret set S consisting of positive integers. You only know that S is finite and non-empty, and satisfies the property that whenever $m, n \in S$, it follows that

$$\frac{m+n}{\gcd(m,n)} \in S.$$

What is the smallest possible number of elements in S?

Answer. 1

Solution. Once you see that $S = \{2\}$ is a possibility, you are done. Indeed this is the only possibility for a set S satisfying these conditions, but you don't need to see all that. The quickest way students will stumble upon this answer is to realize that since S is nonempty, we take any $x \in S$ and see that

$$2 = \frac{x+x}{\gcd(x,x)} \in S$$

Problem 8. Alice finished printing her thesis. It has between 100 and 1,000 pages. To number the pages, starting with 1, she needed twice as many digits as there are pages. How many pages does her thesis have?

Answer. 108

Solution. Let p be the total number of pages. Starting from 100, each page has three digits, i.e. those pages contribute to $3(p-99)$ digits. Below 100, there are 9 single digit pages and 90 double digit pages: these amout to $9 + 2 \cdot 90 = 189$ digits. All in all, for $p \geq 100$, we have $3(p-99) + 189$ digits which yields the equation

$$3(p-99) + 189 = 2p$$

whose solution is 108.

Problem 9. Consider a non-zero polynomial $P(x)$ with real coefficients that satisfies

$$P^2(x) - P^2(y) = P(x+y)P(x-y)$$

for all real numbers x and y. If $P(1) = 6$, what is $P(7)$?

Answer. 42

Solution.

1. **Engineer Solution.** Note that the functional equation above is similar to

$$x^2 - y^2 = (x + y)(x - y)$$

which suggests that $P(x)$ is linear, i.e. $P(x) = ax$. Since $P(1) = 6$, this implies that $P(7) = 42$.

2. **Math Solution.** Letting $x = 0$ we obtain $-P^2(y) = P(y)P(-y)$, i.e. P is an odd polynomial (i.e. all its terms are of odd degree) and in particular P is of degree at least 1. Assume that the leading term of $P(x)$ is ax^n. Let $y = x - 1$ and let's compare the dominating terms of the left and right hand side of the equality. On the left the dominating term has degree $2n - 1$, on the right, the dominating term has degree n. This is inconsistent unless $n = 1$ and $P(x) = ax$. Using the value at 1, we conclude that $P(x) = 6x$ and $P(7) = 42$.

Problem 10. Consider the variables $x_1, x_2, \ldots, x_{2024}$. Determine the number of integer solutions to the equation

$$x_1^{2024} + ... + x_{2024}^{2024} = 20242024,$$

where exactly half the x_i are odd.

Answer. 0

Solution. Note that 2^{2024} is orders of magnitudes larger than the right hand side so all x_i must be take the value 0, 1 or -1. The maximum value of the left hand side is hence bounded by 2024 which is too small, so there are no solutions.