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Ecological utility theory: Solving a series convergence issue

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ABSTRACT

Utility analysis (Patten, 1991; Fath and Patten, 1999) is quite useful in quantifying direct and indirect species relations in a compartmental ecosystems model, regardless of its size or complexity. It serves as a basis for the formulation of system-wide synergism (Fath and Patten, 1998) and mutualism (Fath, 2007) measures. A significant issue that limits the applicability of utility analysis is that its mathematical formulation requires the convergence of a matrix power series, which may fail for otherwise perfectly valid ecosystem models. For example, utility analysis for the well known Neuse river estuary nitrogen flow models (Baird and Ulanowicz, 1989), collected over 4 years (16 seasons total), do not converge for some seasons, but converge for others. Interestingly, ecologists find the analysis results meaningful and useful, even when the convergence criteria are not satisfied. This work investigates the cause for this divergence, analyzes the properties of the matrix power series, and uses an alternative summability method which transforms the diverging matrix power series into a converging one. In particular, we show that finitely many applications of the Euler transform are capable of forcing convergence on an otherwise diverging matrix power series for utility analysis. While the divergence in the regular sense remains, this work brings forward a strong mathematical argument that the utility analysis, synergism and mutualism indices, are useful for all ecological network models, regardless of their convergence characteristics.

1. Introduction

Utility analysis (Patten, 1991; Fath and Patten, 1999) is an invaluable tool to assess the harm or benefit of a species to another, a species to its ecosystem, and the total sum of harm and benefit experienced by the entire ecosystem. The latter system-wide measure is called the synergism index (Fath and Patten, 1998). Utility analysis applies to flow network models of conservative quantities (energy, matter), often depicted as directed graphs. These consist of n compartments (nodes, vertices) interconnected by a set of directed flows (directed links, edges). The compartments denote standing stocks (x) as storages of the energy or matter, such as the total biomass of a certain species living in an area. These quantities are transferred as directed flows (F_{ij}) between compartment pairs.

$$\begin{aligned} F_{ij} &: \text{Flow rate from compartment } j \text{ to compartment } i \\ z_i &: \text{Environmental input rate into compartment } i \\ y_i &: \text{Environmental output rate from compartment } i \\ x_i &: \text{Storage amount at compartment } i \end{aligned} \tag{1}$$

Natural systems are composed of thousands, or even millions, of individuals interacting while the compartments and flows are idealized simplifications of these interactions attempting to model the overall fluxes of the studied quantity between different modes of residence within the system. The network "flows" considered in ecological models are point transfers of mass or energy between the node storages, representing interactions such as feeding among species. The transfer set so constructed represents a system-of-definition, open to energy and matter exchange at the system boundaries, incoming as inputs (z), outgoing as outputs (y). The inputs and storages generate the flows out from a compartment,

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whose sums at ith nodes are the outgoing throughflows (T^{out}).

$$T_i^{out} = y_i + \sum_{j=1}^{n} F_{ji}$$

$$T_i^{in} = z_i + \sum_{j=1}^{n} F_{ij}$$

The total rate of matter or energy received by a compartment defines the incoming throughflow (T_i^{in}). The difference of the incoming and outgoing throughflows defines the change in storage, forming a differential equation

$$\frac{dx_i}{dt} = T_i^{in}(t) - T_i^{out}(t). \quad (2)$$

If the storage values stay constant over time ($dx_i/dt = 0$), meaning that the system is at steady-state, then the incoming and outgoing throughflows for each compartment are equal to each other ($T_i^{in} = T_i^{out} = T_i$). One advantage of utility analysis is that it does not require any information about flow kinetics or dynamics. For an ecosystem model represented as a differential equation (2), it is relatively easy to perform perturbation simulations to measure the effects of a compartment on others. However, deriving an accurate differential equation model of an ecosystem is no easy task, and sometimes not even feasible. Utility analysis quantifies compartmental relationships using only flow rates among compartments and the environment at steady state.

Utility analysis is built on the *direct utility matrix D* (Patten, 1991), defined as

$$D_{ij} = \frac{F_{ij}}{T_i} - \frac{F_{ji}}{T_i} \quad (3)$$

utilizing the steady state assumption that total input into and total output from each compartment equal each other. D_{ij} quantifies the relative benefit ($D_{ij} > 0$) or harm ($D_{ij} < 0$) done by compartment j to compartment i, based on only the direct interactions. For instance, if j consumes i, $F_{ji} > 0$ and clearly compartment j is harmful for i, but the relative intensity of this harm depends on the existence of other consumers of i. For example, if j is the only consumer of i, then $T_i = F_{ji} + y_i$ so $D_{ij} = -F_{ji}/(F_{ji} + y_i)$, is near to -1, indicating that j does a lot of the harm to i. But if compartment i has multiple consumers, T_i will be larger and the relative harm to compartment i done by compartment j will decrease. Hence, the ratio $-F_{ji}/T_i$ represents the relative harm j does to i. Similarly, if i consumes j ($F_{ij} > 0$), then j is beneficial for i. The ratio F_{ij}/T_i represents how beneficial j is for i, among all resources of i. Eq. (3) defines D as a sum of this direct benefit and harm received by i from j, in other words, the direct utility of j for i.

The *utility analysis matrix* U_{ij} quantifies how beneficial ($U_{ij} > 0$) or harmful ($U_{ij} < 0$) compartment j is for i over all possible connections, direct and indirect. Second order effects of j on i are given by the ij entry of the squared matrix, D^2. Indeed, the ij coefficient of D^2 is given by $\left(D^2\right)_{ij} = \sum_k D_{ik}D_{kj}$ with $D_{ik}D_{kj}$ being the product of the relative good (or harm) done by compartment j to compartment k with the relative good (or harm) done by compartment k to compartment i. Summing over all compartments k gives the total second order effects of compartment j on compartment i. Similarly, all nth order effects are given by the elements of the nth power, D^n. Therefore U is defined as a matrix power series of the D matrix, similar to the definitions of pathway, throughflow, and storage analyses (Patten, 1978, 1985; Fath and Patten, 1999):

$$U := I + \underbrace{D}_{\text{Direct}} + \underbrace{D^2 + D^3 + \cdots}_{\text{Indirect}} \quad (4)$$

Table 1
Computations for utility analysis for the two models shown in Fig. 1. The numerical results in this table confirm the visual results presented in Fig. 1(c) and (d). The information on the first column clearly shows that $1 + D + \cdots + D^n$ converges to $(I - D)^{-1}$ as $n \to \infty$ for Model (a), whereas the second column shows that such convergence is not valid for Model (b), and we have $1 + D + D^2 \cdots \neq (I - D)^{-1}$.

	Model (a)	Model (b)
D	$\begin{bmatrix} 0 & -0.51 & -0.24 \\ 1 & 0 & -0.52 \\ 0.48 & 0.52 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -0.59 & -0.09 \\ 1 & 0 & -0.66 \\ 0.34 & 0.66 & 0 \end{bmatrix}$
$\sum_{m=0}^{25} D^m$	$\begin{bmatrix} 0.77 & -0.39 & -0.06 \\ 0.57 & 0.71 & -0.44 \\ 0.51 & 0.26 & 0.85 \end{bmatrix}$	$\begin{bmatrix} 1.59 & -1.05 & -0.75 \\ 2.17 & 1.91 & -1.11 \\ 0.10 & 1.47 & 1.44 \end{bmatrix}$
$\sum_{m=0}^{100} D^m$	$\begin{bmatrix} 0.67 & -0.33 & 0.01 \\ 0.39 & 0.59 & -0.40 \\ 0.52 & 0.15 & 0.80 \end{bmatrix}$	$\begin{bmatrix} 25.9 & 27.2 & -6.9 \\ -29.1 & 39.3 & 32.3 \\ -37.5 & -16.9 & 19.75 \end{bmatrix}$
$(I-D)^{-1}$	$\begin{bmatrix} 0.67 & -0.34 & 0.01 \\ 0.39 & 0.59 & -0.40 \\ 0.53 & 0.15 & 0.79 \end{bmatrix}$	$\begin{bmatrix} 0.68 & -0.35 & 0.09 \\ 0.37 & 0.51 & -0.41 \\ 0.48 & 0.22 & 0.76 \end{bmatrix}$
Eigenvalues of D	$\begin{bmatrix} 0 \\ -0.95i \\ +0.95i \end{bmatrix}$	$\begin{bmatrix} 0 \\ -1.045i \\ +1.045i \end{bmatrix}$

Since $(D^m)_{ij}$ represents the harm and/or benefit received by i from j over all paths of length m, U, defined as the sum of all powers of D, represents the relationship among all compartments, taking into account all direct and indirect connections.

2. Occasional failure of utility analysis computations

A significant problem with the mathematical formulation of the utility matrix (4) limits its use. A necessary condition for the infinite sum of the powers of D to converge to a finite value is that the elements in the infinite sum must become smaller (converge to zero), or at least partially cancel out. In certain cases, the elements of the matrix D^m may alternate between increasingly high positive and negative values as the matrix power m increases, as shown in Fig. 1(d). For those cases the sum defining U diverges. If the infinite sum converges, it must converge to the matrix $(I - D)^{-1}$. This matrix can be constructed regardless of the convergence of the infinite sum (Fath, 2004). It is perhaps tempting to simply define the utility matrix U to be $(I - D)^{-1}$ but then the original motivation of summing all higher order effects is lost.

Indeed, most software performing utility analysis, such as EcoNet (Kazanci, 2007; Schramski et al., 2011), enaR (Borrett and Lau, 2014) and NEA.m (Fath and Borrett, 2006), naturally use $(I - D)^{-1}$ to compute U, as it is not feasible to compute an infinite sum of matrix powers. So the software may display a utility matrix even in the event that the sum defining utility diverges. The equivalence $(I - D)^{-1} = \sum_{m=0}^{\infty} D^m$ relies on an apparently fragile limit operation that may fail for some models. Unfortunately no clear ecological reason has been provided for this failure in the literature so far. For example, the well known Neuse river estuary nitrogen flow models (Baird and Ulanowicz, 1989) contain 16 ecological network models based on data collected for four seasons over four years. The utility analysis matrix converges for some seasons, but not for others. Nevertheless, $(I - D)^{-1}$ can be computed for all seasons, and appears to provide reasonable and meaningful information. Yet, without the necessary convergence, we have no clear explanation as to what the matrix $(I - D)^{-1}$ represents.

To investigate the issue further, we built two similar models with identical network structures, but with slightly different flow values, shown in Fig. 1(a) and (b). Table 1 shows essential matrices computed for these two models. The results shown in Fig. 1(c), (d) and Table 1 clearly demonstrates that the convergence criterion is satisfied by Model (a), but not by Model (b), despite the strong similarity between the two models. Current methodology limits the application of utility analysis to Model (a), and obtained results

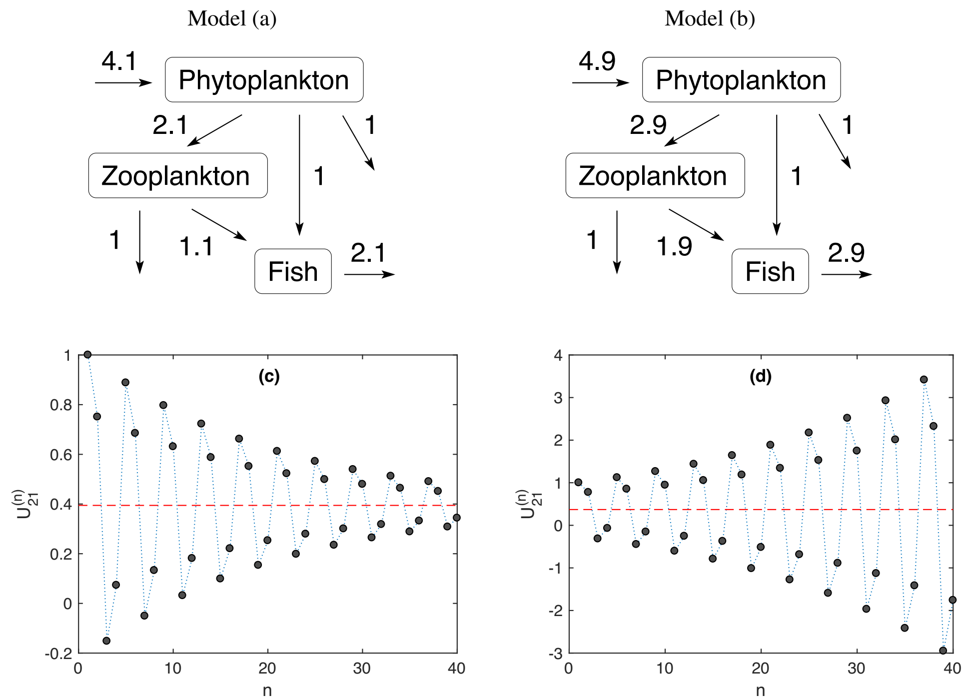

Fig. 1. Two similar ecosystem models are shown. Fig. (c) and (d) shows the partial sum of the powers of D ($U^{(n)} := \sum_{m=0}^{n} D^m$) for the two models, specifically n *vs.* $U_{21}^{(n)}$. The dotted horizontal line in both figures shows $((I-D)^{-1})_{21}$. We chose the entry *21* arbitrarily, since it would be redundant to show the entire matrix, requiring nine very similar figures. Fig. (c) shows that the partial sum $I+D+\cdots+D^n$ converges to $(I-D)^{-1}$ as $n\to\infty$ for Model (a), whereas Fig. (d) shows that this convergence fails for Model (b). Therefore computation of utility analysis fails for Model (b), but not for Model (a), despite their similarities.

$((I-D)^{-1})$ are not, by present theory, valid for Model (b). Nevertheless, two important observations led us to further investigate this issue, and utilize appropriate mathematical methods to overcome this serious limitation:

- The dashed line in Fig. 1(d) passes roughly through the middle of the diverging series, hinting at the possibility that $I+D+D^2+\cdots=(I-D)^{-1}$ might still hold when alternative summability methods are utilized for this mathematically diverging but seemingly "working" series.
- The similarity observed when comparing the matrix $(I-D)^{-1}$ for the two models in Table 1 hints at the usefulness of $(I-D)^{-1}$ even when the convergence fails, as compartments of similar models are likely to have similar utility relations, which happens to be accurately reflected by $(I-D)^{-1}$.

3. Why does the sum of *D* powers diverge?

The mathematical reason for this issue relates to the eigenvalues of D (Patten, 1992). Unfortunately, eigenvalues of D do not appear to be associated with readily available structural or physical properties of an ecosystem model. To pinpoint the cause of divergence, we decompose the D matrix into its eigenvalues and eigenvectors using linear algebra, in particular, spectral theory (Beezer, 2015). Unfortunately, there's little information on the ecological significance, interpretation, or the meaning of the individual components that appear during this analysis. Actually, most of the scalar and matrix values we utilize are not even real, but complex values with non-zero imaginary components. Therefore we primarily focus on the mathematics in order to understand, and then eliminate the convergence issue for the remaining of this manuscript, with little attention given to the ecological significance of the variables or quantities defined during this process.

First, we show that the infinite sum of the powers of the D matrix (4) is equal to $(I-D)^{-1}$, only if the largest magnitude of all the eigenvalues of D is less than 1. We show that D is diagonalizable (Appendix, Corollary 7) for any compartment model representing substance storage and flow, enabling the following representation (Beezer, 2015):

$$D = P\Lambda P^{-1},\ \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Here, Λ is a diagonal matrix of eigenvalues of D, and P is an invertible matrix. Then

$$D^m = \left(P\Lambda P^{-1}\right)^m = P\Lambda^m P^{-1},\quad \Lambda^m = \begin{bmatrix} \lambda_1^m & & \\ & \ddots & \\ & & \lambda_n^m \end{bmatrix}$$

We observe that the size of the elements of D^m is directly related to the size of the elements of Λ^m, as P and P^{-1} are constant matrices (independent of m). In other words, if powers of the eigenvalues of D take large values, so will D, and vice versa. The utility analysis matrix U is defined as:

$$U := \sum_{m=0}^{\infty} D^m = P\left(\sum_{m=0}^{\infty}\Lambda^m\right)P^{-1} = P\begin{bmatrix} \sum_{m=0}^{\infty}\lambda_1^m & & \\ & \ddots & \\ & & \sum_{m=0}^{\infty}\lambda_n^m \end{bmatrix}P^{-1} \tag{5}$$

The convergence of the sum ($\sum_{m=0}^{\infty} D^m$) of powers of D entirely depends on the convergence of the sum of powers of its eigenvalues

($\sum_{m=0}^{\infty}\lambda_k^m$ for all $k=1, \ldots, n$.) In general, for any complex value λ, we have

$$1+\lambda+\lambda^2+\cdots = \lim_{m\to\infty}\frac{1-\lambda^{m+1}}{1-\lambda} = \begin{cases} \dfrac{1}{1-\lambda} & \text{if } \left\|\lambda\right\| < 1 \\ \text{Does not exist} & \text{if } \left\|\lambda\right\| \geq 1 \end{cases}$$

Continuing from Eq. (5), we have that

$$P\begin{bmatrix} \sum_{m=1}^{\infty}\lambda_1^m & & \\ & \ddots & \\ & & \sum_{m=1}^{\infty}\lambda_n^m \end{bmatrix}P^{-1} = P\begin{bmatrix} \frac{1}{1-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{1-\lambda_n} \end{bmatrix}P^{-1}$$
$$= P(I-\Lambda)^{-1}P^{-1}$$
$$= (I-D)^{-1}$$

assuming all eigenvalues of D have magnitude less than one (Patten, 1991; Lobanova et al., 2009). In other words,

$$U=\sum_{m=0}^{\infty}D^m \text{ converges to} (I-D)^{-1} \text{ if and only if } \left\|\lambda_k\right\| < 1$$
for all $k=1, \ldots, n$.

Table 1 shows the eigenvalues of Model (a) and Model (b), illustrating this general result.

4. Eliminating divergence using the Euler transform

What if the magnitude of an eigenvalue of D exceeds one? Then the sum $I+D+D^2+\cdots$ defining U does not converge. As a solution to this problem, we consider an alternative summability method, namely the Euler summation (Hardy, 2000), which provides an alternative method for computing the sum $I+D+D^2+\cdots$ even when the magnitude of an eigenvalue of D exceeds one. The Euler transform of a given series

$$\sum_{n=0}^{\infty}a_n \tag{6}$$

is defined as

$$\sum_{n=0}^{\infty}\frac{A_n}{2^{n+1}}\,A_n=\sum_{k=0}^{n}\binom{n}{k}a_k. \tag{7}$$

The Euler transform conserves the sum of the series. In other words, if the series (6) converges, then so does its Euler transform (7), and to the same sum. On the other hand, it is possible for the Euler transform to converge even when the original series diverges. If the series (7) converges, then the series (6) is called Euler summable. Applying the Euler transform to

$$\sum_{n=0}^{\infty}\lambda^n = 1+\lambda+\lambda^2+\cdots = \frac{1}{1-\lambda} \text{ if } \left\|\lambda\right\| < 1 \tag{8}$$

we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\binom{n}{k}\lambda^k}{2^{n+1}} &= \frac{1}{2}+\frac{1+\lambda}{4}+\frac{1+2\lambda+\lambda^2}{8}+\frac{1+3\lambda+3\lambda^2+\lambda^3}{16}+\ldots \\ &= \frac{1}{2}+\frac{1+\lambda}{4}+\frac{(1+\lambda)^2}{8}+\frac{(1+\lambda)^3}{16}+\ldots \\ &= \frac{1}{2}\left[1+\frac{1+\lambda}{2}+(\frac{1+\lambda}{2})^2+(\frac{1+\lambda}{2})^3+\ldots\right] \\ &= \frac{1}{2}\left[\frac{1}{1-\frac{1+\lambda}{2}}\right] \quad \textit{assuming}\,\|\frac{1+\lambda}{2}\|<1 \\ &= \frac{1}{2}\cdot\frac{2}{2-(1+\lambda)} \\ &= \frac{1}{1-\lambda}\end{aligned} \tag{9}$$

Here, the convergence criterion of the transformed series (9) is $\left\|(1+\lambda/2)\right\| < 1$ whereas the convergence criterion of the original series (8) is $\left\|\lambda\right\| < 1$. Does this help in our case? The ultimate answer is yes, but further work is required to say yes in confidence. Here's a typical application of the Euler transform:

$$\text{Original series} \quad (\lambda=-\frac{1}{2}) \quad 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$$
$$+\cdots=\sum_{n=0}^{\infty}(\frac{-1}{2})^n=\frac{2}{3}$$
$$\text{Euler transform} \quad (\frac{1+\lambda}{2}=\frac{1}{4}) \quad \frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}+\frac{1}{512}$$
$$+\cdots=\frac{1}{2}\sum_{n=0}^{\infty}(\frac{1}{4})^n=\frac{2}{3}$$

Here, both the original series and its Euler transform converge. However, we observe that the convergence of the transform series is much faster, as the individual elements in this series converge to zero much faster than the original series. For $\left\|\lambda\right\| > 1$, the original series will diverge, but its Euler transform may converge. For example:

$$\text{Original series} \quad (\lambda=-2) \quad 1-2+4-8+16-32+\cdots$$
$$=\sum_{n=0}^{\infty}(-2)^n \text{ diverges}$$
$$\text{Euler transform} \quad \left(\frac{1+\lambda}{2}=-\frac{1}{2}\right) \quad \frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}$$
$$+\cdots=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n=\frac{1}{3}$$

While the original series diverges, it is Euler summable, and its Euler sum can be computed using the same formula used for converging series (8), which is not valid for $\left\|\lambda\right\| > 1$.

$$1-2+2^2-2^3+\cdots \stackrel{?}{=} \frac{1}{1-(-2)}=\frac{1}{3}$$

Sometimes a single Euler transformation may not be enough for convergence, but successive applications are required. For example:

$$\begin{array}{llll}
\text{Original series} & (\lambda=-10) & 1-10+100-1000+10^4-10^5+\cdots=\sum_{n=0}^{\infty}(-10)^n & \text{diverges}\\
\text{First Euler transform} & (\frac{1-10}{2}=-\frac{9}{2}) & \frac{1}{2}(1-\frac{9}{2}+(\frac{9}{2})^2-(\frac{9}{2})^3+\cdots)=\frac{1}{2}\sum_{n=0}^{\infty}(-\frac{9}{2})^n & \text{diverges}\\
\text{Second Euler transform} & (\frac{1-9/2}{2}=-\frac{7}{4}) & \frac{1}{4}(1-\frac{7}{4}+(\frac{7}{4})^2-(\frac{7}{4})^3+\cdots)=\frac{1}{4}\sum_{n=0}^{\infty}(-\frac{7}{4})^n & \text{diverges}\\
\text{Third Euler transform} & (\frac{1-7/4}{2}=-\frac{3}{8}) & \frac{1}{8}(1-\frac{3}{8}+(\frac{3}{8})^2-(\frac{3}{8})^3+\cdots)=\frac{1}{8}\sum_{n=0}^{\infty}(-\frac{3}{8})^n=\frac{1}{11} &
\end{array}$$

At this point, we should emphasize that the Euler transformation is not a magic formula that will force any diverging series into a converging sum with enough successive applications. Consider the following example, which has significantly smaller terms than the previous Euler summable series above.

$$\begin{array}{ll}
\text{Original series} & 1+2+4+8+16+32+\cdots=\sum_{n=0}^{\infty}2^n \text{ diverges}\\
\text{Euler transform} & \frac{1}{2}\left(1+\frac{3}{2}+\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^3+\cdots\right)=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n \text{ diverges}
\end{array}$$

This diverging series is not Euler summable. In fact, this series cannot be made to converge with any number of iterations of the Euler transform. Note that all diverging series presented previously had the property that $\lambda \leq -1$. Convergence is already guaranteed for $-1<\lambda<1$. However, a series is not Euler summable if $\lambda>1$. In a sense, the Euler transformation will help provide an "average" value for the alternating series, as shown in Fig. 1(d), which fits perfectly for our purpose. Yet, it is ineffective for diverging non-alternating series, where all terms in the series have the same sign, indicating divergence to either $+\infty$ or $-\infty$.

Now we return to the convergence question for the power series defining utility. As we have seen, the convergence of (4) is determined by the convergence of the geometric sum of the eigenvalues, λ_i. In our case, the eigenvalues are complex numbers. The criterion for Euler summability for sums of powers of complex numbers is that the real part of λ should be less than one. In other words, $1+\lambda+\lambda^2+\cdots$ is Euler summable if $Re(\lambda)<1$. This criterion becomes clear visualizing the disk of convergences for any complex number $\lambda=x+iy$ shown in Fig. 2. The disk of convergence is a set of values for which a series converges, which often forms a circular region.

$$\text{Original series } \left\|\lambda\right\| < 1 \Leftrightarrow x^2+y^2<1 \tag{10}$$

$$\text{First Euler transform } \left\|\frac{1+\lambda}{2}\right\| < 1 \Leftrightarrow (x+1)^2+y^2<4 \tag{11}$$

$$\text{Second Euler transform } \left\|\frac{1+\left(\frac{1+\lambda}{2}\right)}{2}\right\| < 1 \Leftrightarrow (x+3)+y^2<16 \tag{12}$$

In the appendix, we show that eigenvalues of D are purely imaginary or zero (Corollary 6) for any network model. In other words, the real parts of the eigenvalues of D are always zero, satisfying the condition $Re(\lambda)=0<1$ for Euler summability (Fig. 2). This finalizes our proof that $U:=I+D+D^2+\cdots$ is always Euler summable to $(I-D)^{-1}$, for any compartment model, regardless of the values of the eigenvalues of D.

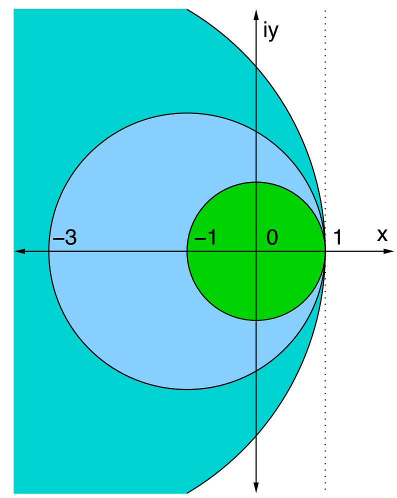

Fig. 2. Figure shows how successive application of Euler transform enlarges the disk of convergence for the series $1+\lambda+\lambda^2+\cdots$ where λ is a complex number. The unit disk satisfies Eq. (10), while the larger disk satisfies Eq. (11), and the largest one satisfies (12). With successive applications of the Euler transform, the disk of convergence will enlarge to cover any value λ if it's real part does not exceed one. In other words, Euler summability is guaranteed for λ provided that $Re(\lambda)<1$ and impossible to achieve if $Re(\lambda)\geq 1$. Since all eigenvalues of D have zero real parts (Corollary 6), Euler summability of the D matrix power series is guaranteed.

5. Conclusion

A major roadblock to the application of utility analysis for complex systems models has been the verification of the convergence of the infinite series that defines the utility matrix. In a striking example of this, Li et al. (2012) treats the city of Beijing as a giant ecosystem, and uses utility analysis to study the relations between the city's main processes such as agriculture, mining, recycling, domestic consumption, processing and manufacturing and construction. The author states "However, the D matrix power series does not always converge, and it is necessary to confirm that the D matrix converges before applying utility analysis." The data set used for their model clearly required a great deal of time and effort to compile. Yet, their manuscript would not be possible if the convergence condition failed

Utility analysis can be viewed as a function f that produces a matrix representing compartment relations based on a systems model.

$$f(F,y,z)=U,\ f:\text{ecosystem model}\longmapsto\text{species relations}$$

If this function fails on some ecosystem models when there's nothing seemingly wrong with the model itself, then a modeler's approach should be to ask if the formulation is too restrictive. To be sure, the field scientists' observations and intuition supersede the insights gained by any mathematical or computational model, as any abstraction of a living system is no longer that living system, but a mathematical or computational construct that may play by its own rules. During a discussion of this convergence issue, Bernard Patten shared his experience that even for ecosystem models where the convergence criterion was not satisfied, $(I-D)^{-1}$ appeared to provide reasonably correct information. This conversation led us to investigate the original definition of utility analysis further, as opposed to searching for an alternative or a revision.

Despite its rather technical character, the application of this work is transparent in use, as it does not develop a new methodology or construct, but removes an existing obstacle in current methodology. It is a significant achievement in enabling widespread adoption of utility analysis for ecosystem models, as it eliminates the strict requirement that norms of all eigenvalues of D should be less than one.

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Appendix A. *D* is diagonalizable, and its eigenvalues are purely imaginary or zero

The proof follows from the fact that D is always skew-symmetrizable, and such matrices have purely imaginary eigenvalues. We provide a detailed proof requiring only basic linear algebra knowledge (Beezer, 2015).

Definition 1. A real matrix A is called skew-symmetric (or anti-symmetric) if $A_{ij} = -A_{ji}$ for all i, j, or, in other words, $A = -A^T$.

Theorem 2. *Eigenvalues of skew-symmetric matrices are purely imaginary.*

Proof. Let λ be an eigenvalue of a skew-symmetric matrix A with the associated eigenvector v.

$$Av = \lambda v \quad \Rightarrow \quad \bar{v}^T Av = \bar{v}^T \lambda v = \lambda \|v\|^2$$

Here, $\bar{v}$ represents the complex conjugate of v, and below such conjugates generally. Similarly

$$\bar{v}^T Av = (A^T \bar{v})^T v = (-A\bar{v})^T v = (-\bar{A}\bar{v})^T v = -(\bar{\lambda}\bar{v})^T v$$
$$= -\bar{v}^T \bar{\lambda} v = -\bar{\lambda}\|v\|^2$$

Since v is an eigenvector, $\|v\| \neq 0$, so we get

$$\lambda = -\bar{\lambda}$$

which implies that real part of λ is zero. Therefore λ is either purely imaginary or zero. □

Definition 3. Two matrices A and B are called similar if there exists an invertible matrix P such that $A = P^{-1}BP$.

Theorem 4. *D is always similar to a skew-symmetric matrix.*

Proof. We can define D using matrix definition

$$D = \mathbb{T}^{-1}(F - F^T) \text{ where } \mathbb{T} = \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_n \end{bmatrix}$$

where R is defined as a diagonal throughflow matrix. Then

$$D = \mathbb{T}^{-1}(F - F^T) = \mathbb{T}^{-1/2} \underbrace{\left(\mathbb{T}^{-1/2}(F - F^T)\mathbb{T}^{-1/2}\right)}_{=H} \mathbb{T}^{1/2}$$

This shows that D is similar to a skew-symmetric matrix if $H = \mathbb{T}^{-1/2}(F - F^T)\mathbb{T}^{-1/2}$ is a skew-symmetric matrix:

$$H^T = (\mathbb{T}^{-1/2}(F - F^T)\mathbb{T}^{-1/2})^T = \mathbb{T}^{-1/2}(F^T - F)\mathbb{T}^{-1/2}$$
$$= -(\mathbb{T}^{-1/2}(F - F^T)\mathbb{T}) = -H$$

□

Theorem 5. *Similar matrices have the same eigenvalues.*

Proof. Let A and B similar matrices, and let λ be an eigenvalue of A.

$$Av = \lambda v$$
$$P^{-1}BPv = \lambda v$$
$$B(Pv) = \lambda(Pv)$$

Then λ is an eigenvalue of B as well, with eigenvector Pv. □

Corollary 6. *Combining the results of all theorems shows that the eigenvalues of D are either zero or purely imaginary. In other words, the real parts of the eigenvalues of D are always zero.*

Corollary 7. *D is always diagonalizable.*

Proof. We showed that D is similar to a skew-symmetric matrix. Real skew-symmetric matrices are normal matrices ($A^TA = AA^T$) and are thus subject to the spectral theorem (Beezer, 2015), which states that they can be diagonalized by a unitary matrix. □

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