Complex Analysis Qualifying Exam – Fall 2024

All problems are of equal weight. Please arrange your solutions in numerical order even if you do not solve them in that order. Show work and carefully justify/prove your assertions.

1. Suppose that f is an entire function and that for every $z_0 \in \mathbb{C}$, in the power series
$$f(z) = \sum_{n=0}^{\infty} a_n(z_0)(z - z_0)^n,$$
there is $n \geq 0$ such that $a_n(z_0) = 0$. Prove that f is a polynomial.

2. For any integer $N \geq 2$, compute
$$\int_0^\infty \frac{dx}{1 + x^N}.$$
[Hint: It is convenient to compute an integral along the path from 0 to R, from R to $Re^{2\pi i/N}$ and finally from $Re^{2\pi i/N}$ to 0.]

3. Let $f(z)$ be holomorphic on a region Ω containing the closed unit disc. Suppose:
(1) $f(0) = 0$;
(2) $f(iz) = f(z)$ for all $z \in \Omega$; and
(3) $|f(z)| \leq 2024$ for all $|z| < 1$.
Prove that $|f(1/7)| < 1$.
[Hint: consider the power series expansion of f at 0.]

4. Let $\alpha \in \mathbb{C}$, $|\alpha| = 1$.
(a) Find the number of solutions of the equation $\sin z = \frac{\alpha}{z^2}$ in the strip $\frac{\pi}{2} < \mathrm{Re}(z) < \frac{3\pi}{2}$.
(b) Find the number of solutions to the above equations in the strip $|\mathrm{Re}(z)| < \frac{\pi}{2}$.
[Hint: Estimate $|\sin z|$ from below when $\mathrm{Re}(z) = k\pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$, and use Rouché's Theorem.]

5. Find an explicit conformal mapping ϕ from the strip $S := \{|\mathrm{Im}(z)| < 1\}$ onto the region $\Omega := \{|z - 3/4| > 1/4\} \cap \{\mathrm{Re}(z) < 1\}$.

6. Let $f(z)$ be an entire function. Assume there are constants $M, R > 0$ and an integer $n > 0$ such that $|f(z)| \geq M|z|^n$ for all $|z| > R$. Prove that f is a polynomial of degree at least n.

7. Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function with compact support such that for every integer $n \geq 0$ we have
$$\int_{\mathbb{R}} f(x)x^n e^{-x^2}\, dx = 0.$$
Prove that $f = 0$.
[Hint: Consider the function $F : \mathbb{C} \to \mathbb{C}$ defined by $F(z) = \int_{\mathbb{R}} f(x)e^{zx - x^2}\, dx$ for $z \in \mathbb{C}$ and use the fact that $g \equiv 0$ if and only if $\hat{g} \equiv 0$, where $\hat{g}(\xi) = \int_{\mathbb{R}} g(x)e^{-2\pi i x\xi}\, dx$.]